MATHEMATICS MAGAZINE

CONTENTS

VOLUME 44 · SEPTEMBER 1971 · NUMBER 4

MATHEMATICS MAGAZINE

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ON THE TEACHING OF MATHEMATICS

We . . . record our disagreement with the large number of public personalities at the present time who demand of scientists in general and mathematicians in particular that they should devote their energies to producing the legions of technologists whose existence is, it appears, urgently indispensable to our survival. Things being as they are, it seems to us that in the scientifically and technologically overdeveloped "great" nations in which we live, the first duty of the mathematician—and of many others—is to produce what is not demanded of him, namely, men who are capable of thinking for themselves, of unmasking false arguments and ambiguous phrases, and to whom the dissemination of truth is infinitely more important than, for example, world-wide three-dimensional color T.V.—free men, and not robots ruled by technocrats. It is sad but true that the best way of producing such men does not consist in teaching them mathematics and physical science; for these are branches of knowledge which ignore the very existence of human problems, and it is a disturbing thought that our most highly civilized societies accord them the first place. But even in the teaching of mathematics it is at least possible to attempt to impart a taste for freedom and reason, and to accustom the young to being treated as human beings endowed with the faculty of reason.—Roger Godement in the preface to *Algebra*, Houghton Mifflin, Boston, and Hermann, Paris, 1968.

RECTIFIABLE CURVES ARE OF ZERO CONTENT

R. B. BURCKEL, Kansas State University and
C. GOFFMAN, Purdue University

Our purpose is to give a simple proof of the fact that the image of a rectifiable curve in the plane has 2–dimensional content zero, i.e., can be covered by finitely many squares the sum of whose areas is arbitrarily small.

Let $I=[0, 1]$ and let f be a continuous function on I into the plane. The image $f(I)$ is compact and so it is contained in a closed square Q.

For each partition $\pi=\{t_0=0<t_1< \cdots <t_n=1\}$ of I let

$$l(\pi,f) = \sum_{i=1}^{n} |f(t_i),f(t_{i-1})|,$$

where $|p, q|$ is the distance between points p and q in the plane. Rectifiability means

$$L(f) = \sup_{\pi} l(\pi,f) < \infty.$$

Suppose now that f is rectifiable, $f(I)\subset Q$, and the sides of Q are of length 2. We partition Q into a set E_n of $4n^2$ nonoverlapping squares whose sides are of length $1/n$. The set E_n of squares may be split into 4 pairwise disjoint sets A_n,

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B_n, C_n, D_n each containing n^2 squares such that, for any distinct squares Q_1, Q_2 belonging to the same set, $p\in Q_1$, $q\in Q_2$ implies $|p, q|\geqq 1/n$.

Suppose for some n, $f(I)$ meets at least $n\sqrt{n}$ distinct squares in A_n. Then $L(f)\geqq n\sqrt{n}\cdot 1/n=\sqrt{n}$. So, there is an n_0 such that $f(I)$ meets fewer than $n\sqrt{n}$ squares of A_n and similarly of B_n, C_n, and D_n, whenever $n>n_0$. Accordingly, $f(I)$ meets fewer than $4n\sqrt{n}$ squares of E_n for every $n>n_0$. The content of the union of those squares in E_n which meet $f(I)$ is less than $(4n\sqrt{n})/n^2=4/\sqrt{n}$ so that the content of $f(I)$ is zero.

A FRESH LOOK AT GEOMETRY

V. W. BRYANT, University of Sheffield, England

For a long time geometry has been a little neglected as a subject of higher study in pure mathematics, but here I hope to illustrate one aspect of some fresh axiomatic approaches which might be the seed of renewed interest in the subject.

Let X be a nonempty set and for each a, $b\in X$ let ab denote some subset of X. For example X might be the Euclidean plane considered as a set of points and ab might denote the closed line segment joining a and b. Now if A, $B\subset X$ and $a\in X$, then we define AB, aB, Ba by

$$AB = \bigcup_{a\in A, b\in B} ab, \qquad aB = \{a\}B, \qquad Ba = B\{a\}.$$

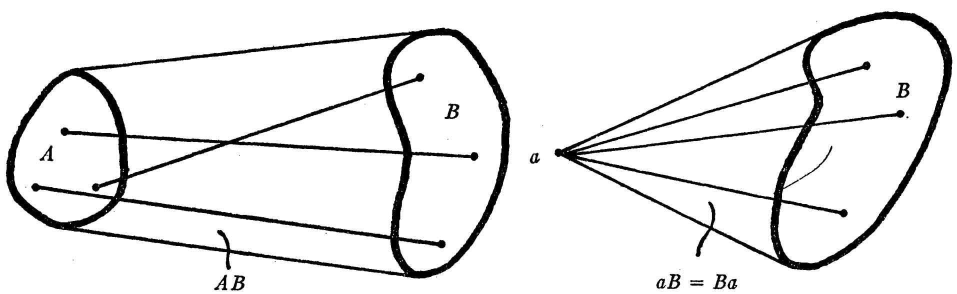

FIG. 1.

In Figure 1 we illustrate AB, aB and Ba in the case of the Euclidean plane, and in this example the latter two coincide. In particular, if a, b, $c\in X$, then ab, bc are subsets of X and $a(bc)$, $(ab)c$ are well-defined subsets of X. If for each a, b, $c\in X$ we have $a(bc)=(ab)c$, then we shall call X an *associative geometry*. Let us see what this means in the case of the Euclidean plane. In Figure 2 we illustrate the two sets $a(bc)$, $(ab)c$ and observe that, though the method of construction is different in each case, the sets coincide and they are just the closed triangular region with vertices a, b, c.

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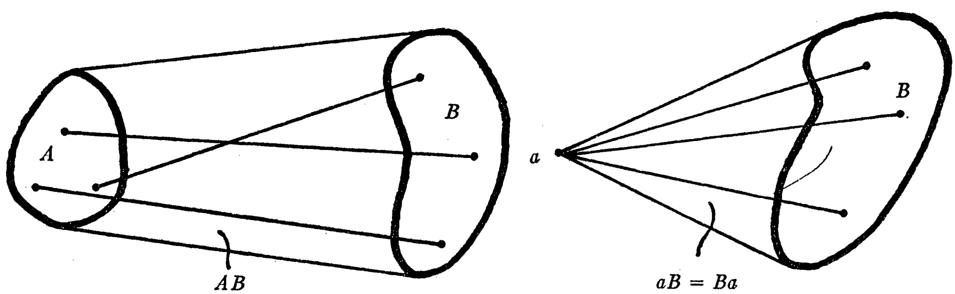

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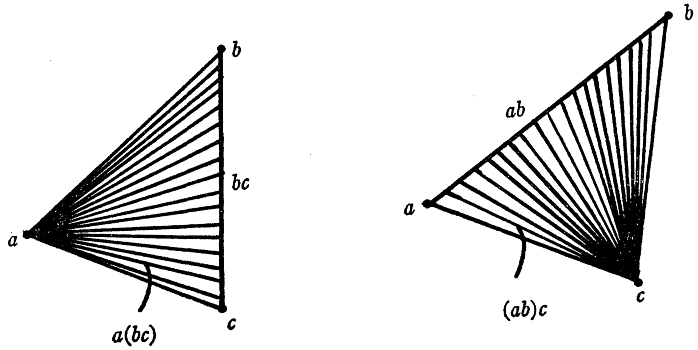

FIG. 2.

Hence the Euclidean plane is an associative geometry, as is any vector space with the subset ab given by

$$ab = \{\lambda a + (1 - \lambda)b : 0 \leqq \lambda \leqq 1\}.$$

It is easy to see that the associative law $a(bc) = (ab)c$ *extends to subsets* A, B, C of X to give $A(BC) = (AB)C$. For example, if X is an associative geometry, a, $b \in X$ and $C \subset X$, then

$$(ab)C = \bigcup_{c \in C} (ab)c = \bigcup_{c \in C} a(bc) = a\left(\bigcup_{c \in C} bc\right) = a(bC).$$

A set $A \subset X$ is *star-shaped* from $a \in A$ if whenever $b \in A$ it follows that $ab \subset A$ (or, more succinctly, if $aA \subset A$). The set of points from which A is star-shaped is called the *kernel* of A and denoted by A^*. In the case of the Euclidean plane this is illustrated in Figure 3. Finally $A \subset X$ is *convex* if whenever a, $b \in A$ it follows that $ab \subset A$ (or, more succinctly, if $AA \subset A$). This is illustrated in Figure 4. Observe also that in Figure 3 A^* is convex. These definitions coincide with the traditional ones in a vector space.

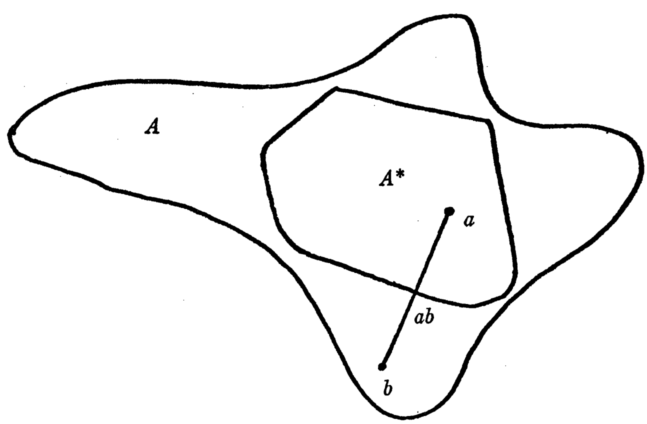

FIG. 3.

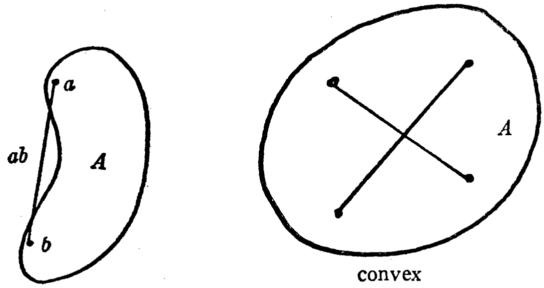

FIG. 4.

In 1912 Brunn, in [1], proved at length that the kernel of any set in the Euclidean plane is convex. Since then this result has been extended to any vector space (see, for example, [3] p. 5). We now generalize this result further to any associative geometry, and in this more general form the proof of the theorem is surprisingly simple.

THEOREM (Brunn). *If X is an associative geometry and $A\subset X$, then A^* is convex.*

Proof. If $A\subset X$, then to show that A^* is convex we take $a, b\in A^*$, $c\in ab$ and show that $c\in A^*$. But $aA\subset A$, $bA\subset A$ and so

$$cA \subset (ab)A = a(bA) \subset aA \subset A.$$

Thus $c\in A^*$ which shows that $ab\subset A^*$, and A^* is convex as required.

In [2] Prenowitz outlines a whole system of axiomatic geometry; I have merely chosen one such axiom and illustrated a practical application which at one and the same time generalizes and simplifies the classical approach. My only hope is that it shows a part of the exciting world of geometry which is still very much alive.

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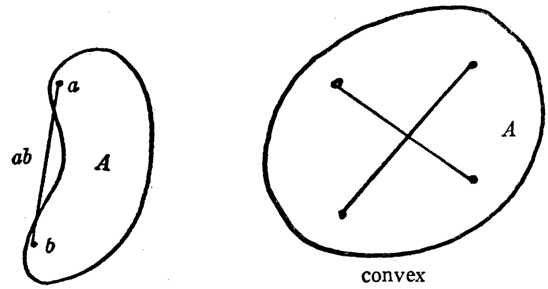
1. H. Brunn, Über Kerneigebiete, Math. Ann., 73 (1912) 436–440.
2. W. Prenowitz, A contemporary approach to classical geometry, Amer. Math. Monthly, 68 (1961), 1–67.
3. F. Valentine, Convex Sets, McGraw-Hill, New York, 1964.

COMPLETION OF A METRIC SPACE

GEORGE R. SELL, University of Minnesota

1. Introduction. The standard way of completing a metric space is to imbed the given metric space in another space consisting of equivalence classes of Cauchy sequences, (cf. [1] pp. 54–58). While this method is very simple to describe, the technical problems arising in the proof require the patience of Job to solve.



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If one is willing to accept the fact that the real line R (with the usual metric) is complete, a simpler proof is possible, (cf. [4]). We present it here.

The motivation for our proof lies in the theory of functional analysis. That is, in order to show that a normed linear space X has a completion one can imbed X into its second conjugate X^{**} by means of the natural imbedding. By using the Hahn-Banach theorem one then shows that this imbedding is an isometry.

Our argument develops along those lines but it is not necessary to be familiar with second conjugate spaces to understand it. We start with a metric space X and construct an appropriate "second conjugate" LC^*. Then we imbed X in LC^* by means of an isometry Φ. Since LC^* is complete, it follows that the closure of $\Phi(X)$ is complete. Of course, since we do not assume X to be a linear space we cannot expect to prove that the mapping Φ is an isomorphism. However, if X is a normed linear space it will be apparent how our argument can be modified. Furthermore, it will also become apparent, from this point of view, why the Hahn-Banach theorem (or something equivalent to it) is needed in order to show that the natural imbedding of a normed linear space into its second conjugate is an isometry.

The idea of defining the conjugate and second conjugate for a metric space, which does not have a linear structure, seems to be new. This concept may be useful in other contexts. For example, one of the consequences of this note is that every F-space [3, p. 51] is isometric with a subset of a Banach space.

Even though our proof is motivated by the theory of functional analysis, we use only two very elementary facts from this theory, *viz.*,

1. the concept of the conjugate of a normed linear space, and
2. the fact that the conjugate of a normed linear space is complete, (cf. [3] p. 62).

2. Completion theorem. First we recall a few definitions. A mapping $\Phi: X \rightarrow Y$ is said to be an *isometry* if it preserves distances, where X and Y are metric spaces. If X and Y are linear spaces then an isometry Φ is an *isomorphism* if it is linear. A metric space Y is said to be a *completion* of X if Y is complete and there is an isometry $\Phi: X \rightarrow Y$ with the property that $\Phi(X)$ is dense in Y. It is clear that if $\Phi: X \rightarrow Y$ is an isometry and if Y is complete, then the closure of $\overline{\Phi(X)}$ is a completion of X.

Let X be a metric space with metric d. Let LC denote the space of all real-valued Lipschitz continuous functions defined on X and vanishing at x_0. That is, $f \in LC$ if and only if $f(x_0) = 0$ and

$$|f(x) - f(y)| \leqq kd(x, y), \qquad (x, y \in X). \tag{1}$$

The infimum of all k that satisfy (1) will be denoted by $\|f\|$. It is clear that $(LC, \|\cdot\|)$ is a normed linear space. (If we drop the condition that $f(x_0) = 0$, then $\|\cdot\|$ is a pseudo-norm.) Let LC^* be the conjugate space of LC; that is LC^* consists of all bounded linear functionals on LC. The norm on LC^* is given by

$$\|l\| = \sup\{ |l(f)| : \|f\| \leqq 1\}.$$

It can easily be shown (cf. [3] p. 62) that LC^* is a complete normed linear space.

We define a mapping $\Phi: X \rightarrow LC^*$ by

$$\Phi: x \rightarrow \delta_x \tag{2}$$

where $\delta_x(f) = f(x)$. One then has

$$\begin{aligned} \|\delta_x - \delta_y\| &= \sup\{ | \delta_x(f) - \delta_y(f) | : \|f\| \leqq 1\} \\ &= \sup\{ | f(x) - f(y) | : \|f\| \leqq 1\} \\ &\leqq \sup\{ \|f\| d(x, y) : \|f\| \leqq 1\} = d(x, y). \end{aligned} \tag{3}$$

In order to show that Φ is an isometry we must find a function f in LC with the property that

$$\|f\| \leqq 1 \quad \text{and} \quad | \delta_x(f) - \delta_y(f) | = d(x, y). \tag{4}$$

(This function may depend on x and y.) However, if we choose

$$f(t) = \tfrac{1}{2}[d(x, t) - d(y, t)] + C, \tag{5}$$

where C is chosen so that $f(x_0) = 0$, we see that this f satisfies (4).
It follows then that the closure of $\Phi(X)$ in LC^* is a completion of X.

One can easily show, using this approach, that any two completions of a metric space X are isometrically equivalent. We omit these details.

3. Completion theorem, linear version. It is evident that the mapping Φ in (2) is not linear even when X is a linear space. The reason for this is that there are functions f in LC that are nonlinear. In fact, the function in (5) is such an example. However, if X is a normed linear space one can replace LC with X^*, the first conjugate of X, and LC^* by X^{**}, the second conjugate. The mapping Φ in (2) is then linear and (3) is proved as above. One now wants to prove (4) which says that the linear functional $\delta_x - \delta_y$ assumes its maximum on the unit ball in X^*. Since the function in (5) is not in X^* (when $x \neq y$) one must find another maximum point, and this is where the Hahn-Banach theorem is used, (cf. [1] pp. 203–204 and [3] p. 66).

Finally we note that another proof of completion is known (cf. [2]), where one imbeds X isometrically into the space of real-valued, bounded continuous functions on X. While this third method is similar to ours, it does not have any apparent modification for the case where X is a normed linear space.

The author wishes to thank Professor A. W. Naylor for some helpful discussions.
This work was partially supported by NSF Grant No. GP-7041X.

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SOME GENERALIZATIONS OF THE 14–15 PUZZLE

HANS LIEBECK, University of Keele, Staffordshire, England

1. Introduction. Sam Loyd's well known "14–15 puzzle" has now enjoyed popularity for almost a century. The puzzle consists of a 4×4 square box containing fifteen equal square unit counters numbered $1, 2, 3, \cdots, 15$. The counters are arranged as in Figure 1 with all but the last two numbers in serial order. The problem is to interchange the counters labelled 14 and 15, thus completing the serial ordering. Sliding movements only are permitted and no counter may be lifted from the box. It is well known that the puzzle is insoluble.

A number of variations of the puzzle have been suggested:

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

FIG. 1.

The rectangular box puzzle. An $m\times n$ rectangular box contains $mn-1$ square counters in m rows and n columns, with the bottom right hand square vacant. The problem is to determine which rearrangements of the counters can be obtained by sliding only, such that in the final position the vacant square again appears in the bottom right hand corner.

The quarter-turn puzzle. The counters in the $m\times n$ box are presented in serial order, reading along rows. The box is given a quarter-turn. The counters should now be rearranged in serial order, again reading along rows. This puzzle is mentioned in W. W. R. Ball [1, Chapter XI] and Martin Gardner [2, p. 20]. Ball claims incorrectly that the puzzle is insoluble for an $m\times n$ box when m and n are both even. Indeed, it is easy to see that for $m=n=2$ the three counters *can* be restored to serial order after a quarter-turn. However, for a 2×4 or 4×4 box the puzzle is insoluble.

More interesting is Ball's claim that for $m=3$, $n=5$ the quarter-turn puzzle is insoluble. His claim is correct for a clockwise quarter-turn but incorrect for a counterclockwise quarter-turn!

The half-turn puzzle. As above, with a half-turn replacing a quarter-turn.

The row-to-column puzzle. Starting with the counters in serial order reading along rows, rearrange them to appear serially in columns, both readings to start in the top left hand corner.

This note is devoted to an analysis of these puzzles. My interest in the puzzles was aroused by a stimulating talk on mathematical toys given to our student Mathematics Society by Professor Walter Ledermann, of the University of Sussex, and I wish to record my thanks.

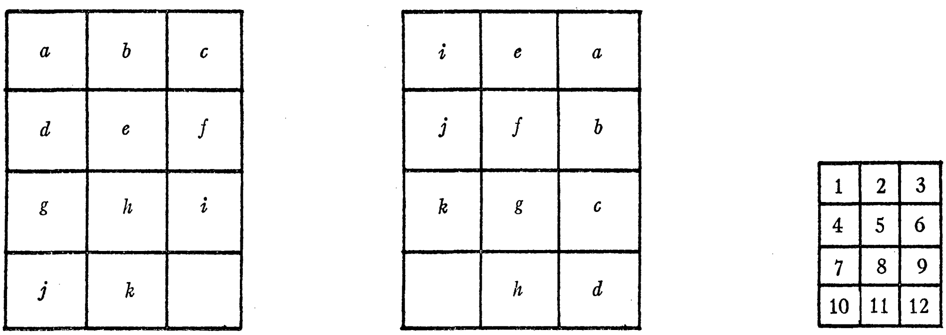
2. The rectangular box puzzle. Consider some arrangement of $mn-1$ labelled counters in the $m\times n$ box. An arbitrary rearrangement may be represented by a permutation as follows: label the mn spaces of the box that the counters can occupy by the numerals $1, 2, 3, \cdots, mn$, reading from left to right along rows, with 1 in the top left hand corner and mn in the bottom right hand corner. Relative to this fixed frame of reference a rearrangement may be represented uniquely by a permutation π on the symbols $1, 2, \cdots, mn$, where $\pi(i)$ denotes the final position of the counter or empty space that prior to the rearrangement was in position i.

In order to avoid confusion with our fixed reference system we shall not label the counters by numerals. Instead we label them alphabetically: $a, b, c, \cdots$.

In Figure 2 we illustrate the case $m=4$, $n=3$. The rearrangement from position (i) to position (ii) is represented by the permutation

$$\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 3 & 6 & 9 & 12 & 2 & 5 & 8 & 11 & 1 & 4 & 7 & 10 \end{pmatrix}. \tag{1}$$

The *inverse rearrangement* from position (ii) to position (i) is represented by π_1^{-1}.



(i) (ii) *Reference frame*

FIG. 2.

A sequence of rearrangements represented in order by permutations π_1, $\pi_2, \cdots, \pi_r$ is equivalent to the single rearrangement represented by the product $\pi_r \cdots \pi_2\pi_1$. (We multiply permutations from right to left.)

A rearrangement of the counters that can be effected by sliding will be called a *slide*. Clearly the collection of all permutations representing slides forms a subgroup of the symmetric group of degree mn.

A single slide of one counter is called a *simple* slide. Such a move is represented by a transposition (but not conversely). For example, in Figure 2 (ii) the slide of the counter labelled "k" into square 10 is represented by the transposition (7 10), in cycle notation.

We say that the counters are in a *standard* position if the bottom right hand square is vacant. Thus in Figure 2, (i) is a standard position, but (ii) is not. If the permutation π represents a rearrangement from one standard position to another, then $\pi(mn)=mn$.

THEOREM 1. *For the $m\times n$ box, a slide from one standard position to another is represented by an even permutation leaving the symbol mn fixed. Conversely, every such permutation represents a slide.*

The first half of this theorem is proved in Ball [**1**]. The converse, which is assumed without proof in many puzzle books, is proved in Story [**3**], where a somewhat lengthy discussion appears. More recently, Spitznagel [**4**] gave a proof using modern methods. In this paper, we omit the proof.

COROLLARY 1. *Let π be the permutation of a rearrangement such that in the initial position the vacant square is in the r_1th row and c_1th column, and in the final position it is in the r_2th row and c_2th column. Then π represents a slide if and only if it is an even or odd permutation according as $r_1+r_2+c_1+c_2$ is even or odd.*

Proof. The initial and final positions may be changed into standard positions by a sequence of $(m-r_1)+(n-c_1)$ and $(m-r_2)+(n-c_2)$ simple slides respectively. The result now follows from Theorem 1.

3. The quarter-turn puzzle. The solution depends on the direction of the quarter-turn. Let us consider the problem for a counterclockwise turn. Figure 2 illustrates the puzzle for a 4×3 box: this puzzle is solvable if and only if the permutation π_1 defined by (1) represents a slide. In general, the $m\times n$ (counterclockwise) quarter-turn puzzle is solvable if and only if the permutation

$$(2)\quad \pi=\begin{pmatrix}1 & 2 & \cdots & m & \cdots & (n-r)m+1 & (n-r)m+2 & \cdots & (n-r+1)m & \cdots & (n-1)m+1 & \cdots & nm\\ n & 2n & \cdots & mn & \cdots & r & r+n & \cdots & r+(m-1)n & \cdots & 1 & \cdots & 1+(m-1)n\end{pmatrix}$$

represents a slide. Note that π effects a rearrangement sending the vacant square from the bottom right hand to the bottom left hand position (of the fixed box). By Corollary 1 we have

LEMMA 1. *The counterclockwise quarter-turn puzzle is solvable if and only if the permutation π defined by* (2) *is even for odd n and odd for even n.*

Next we consider the parity of π.

LEMMA 2. *The permutation π is expressible as a product of $\frac{1}{4}mn(m+1)(n-1)$ transpositions.*

Proof. Consider the sequence

$$n,\ 2n,\ 3n,\cdots,mn,\quad n-1,\ 2n-1,\cdots,mn-1,\cdots,r,$$
$$r+n,\cdots,r+(m-1)n,\cdots,1,\ 1+n,\cdots,1+(m-1)n.$$

(This is the lower row in the definition (2) of π.) We count the number of inversions required to transform it into serial order 1, 2, 3, $\cdots$, mn. First, 1 is brought into the first position by $m(n-1)$ inversions. Next, 2, $\cdots$, n require respectively $m(n-2)$, $m(n-3)$, $\cdots$, m, 0 inversions to bring them into serial position. The first n integers are now in place. To bring the next integer, $1+n$, into position it must be passed over $(n-1)$ blocks of $(m-1)$ symbols. Proceeding in this way, we obtain the number of inversions for the serial order as

$$\begin{aligned}&[m(n-1)+m(n-2)+\cdots+m]\\ &\qquad+[(m-1)(n-1)+(m-1)(n-2)+\cdots+(m-1)]+\cdots\\ &\qquad+[(n-1)+(n-2)+\cdots+1]\\ &\quad=(m+(m-1)+\cdots+1)((n-1)+(n-2)+\cdots+1)\\ &\quad=\tfrac{1}{2}m(m+1)\cdot\tfrac{1}{2}n(n-1).\end{aligned}$$

THEOREM 2. *The (counterclockwise) quarter-turn puzzle is solvable for the following values of m and n and for no others.*

m (rows)	*n (columns)*
Any	$4l+1$
$4k+1$, $4k+2$	$4l+2$
$4k+3$, $4k$	$4l+3$

Proof. This follows from Lemmas 1 and 2. We end this section with a few remarks.

1. The puzzle is not solvable for a box whose number of columns is a multiple of 4.

2. The *clockwise* quarter-turn puzzle for an $m\times n$ box is solvable if and only if the counterclockwise puzzle is solvable for an $n\times m$ box.

Thus for example the 4×3 puzzle is solvable clockwise, but not counterclockwise. This may be made the basis for a mathematical puzzle involving counters having quarter-turn symmetry.

4. The half-turn puzzle. The $m\times n$ box with its counters in alphabetical order is given a half-turn, and the counters are rearranged into alphabetical order (reading from top left along rows in each case). By Lemma 2, the corresponding permutation π' is expressible as a product of $\frac{1}{4}mn(m+1)(n-1)+\frac{1}{4}nm(n+1)(m-1)=\frac{1}{2}mn(mn-1)$ transpositions. (Proof: consider the half-turn as a combination of two quarter-turns.)

Applying Corollary 1, we obtain

THEOREM 3. *The half-turn puzzle is solvable except for the following pairs* (m, n): $(4k, odd)$, $(odd, 4l)$, $(4k+1, 4l+3)$, $(4k+3, 4l+1)$.

This result suggests the following card trick: on a tray of appropriate size, lay out in a 4×3 array the Ace, 2, 3, $\cdots$, 9, 10, Jack of diamonds, leaving the bottom right hand square blank. Destroy the order of the cards by random sliding within the 4×3 array. Now hand the tray to your victim, *giving it a half-turn as you do so*, and ask him to restore the serial order. He will of course find this feat impossible. The trick should be performed with diamonds, as these cards possess half-turn symmetry.

5. The row-to-column puzzle. This may be considered by applying column permutations to the quarter-turn puzzle. Specifically, the puzzle is solved by a permutation π_2 that is the product of two permutations $\pi_2=\pi_3\pi$. Here π is the permutation defined by (2), and π_3 is the permutation defined by $\pi_3(nr+j)=(r+1)n-j+1$, for $r=0, 1, \cdots, m-1$; $j=1, \cdots, n$. The permutation π_3 represents a rearrangement that reverses the order of counters in each of the m rows. In each row, this rearrangement can be realized by $(n-1)+(n-2)+\cdots+1$ transpositions of counters. Hence π_3 is the product of $\frac{1}{2}mn(n-1)$ transpositions. And π_2 is the product of $\frac{1}{4}mn(m+1)(n-1)+\frac{1}{2}mn(n-1)=\frac{1}{4}mn(n-1)(m+3)$ transpositions. It will be found that the puzzle is solvable except for the following pairs (m, n): $(4k+2, 4l+2)$, $(4k+3, 4l+3)$, $(4k+2, 4l+3)$, $(4k+3, 4l+2)$.

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1. W. W. R. Ball, Mathematical Recreations and Essays, Macmillan, New York, 1939.
2. Martin Gardner, Mathematical Puzzles of Sam Loyd, Dover, New York, 1959.
3. W. E. Story, Note on the "15" puzzle, Amer. J. Math., 2 (1879) 399–404.
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THE EXISTENCE OF THE DERIVATIVE OF THE INVERSE FUNCTION

NORTON STARR, Amherst College

In [2] an example is given of a one-to-one function f mapping $(0, 7/2)$ onto $(-1, 4)$ and such that $f'(1)=2$ yet the inverse of f is not only nondifferentiable but even discontinuous at $f(1)$. The following example exhibiting similar behavior has been of use to the author in teaching calculus and seems somewhat simpler.

Let $A=\{1/n \mid n=1, 2, \cdots\}$ and $B=\{1+1/n \mid n=1, 2, \cdots\}$. Let

$$f(x)=\begin{cases} x & x\notin A\cup B. \\ 1/(2^n+n+k) & x=1/(2^n+k) \qquad n=0,1,2,\cdots;\quad 0\leqq k<2^n \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{i.e., } x\in A. \\ 1/(2^{n+1}+n) & x=1+1/(n+1) \qquad n=0,1,2,\cdots \qquad \text{i.e., } x\in B. \end{cases}$$

To determine the image of $(-\infty, \infty)$ under f, consider the images $f(A)$ and $f(B)$ of A and B. It is evident that $f(A)\subset A$ and that the only elements of $A-f(A)$

lay out in a 4×3 array the Ace, 2, 3, $\cdots$, 9, 10, Jack of diamonds, leaving the bottom right hand square blank. Destroy the order of the cards by random sliding within the 4×3 array. Now hand the tray to your victim, *giving it a half-turn as you do so*, and ask him to restore the serial order. He will of course find this feat impossible. The trick should be performed with diamonds, as these cards possess half-turn symmetry.

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To determine the image of $(-\infty, \infty)$ under f, consider the images $f(A)$ and $f(B)$ of A and B. It is evident that $f(A)\subset A$ and that the only elements of $A-f(A)$

are those of the form $1/(2^n+n+2^n)=1/(2^{n+1}+n)$ $(n=0, 1, 2, \cdots)$. But these elements are just the image under f of the set B. Thus A is the disjoint union of $f(A)$ and $f(B)$. Hence the image of $(-\infty, \infty)$ under f is $(-\infty, \infty)-B$. f is readily seen to be one-to-one in turn on A, B, $A\cup B$, and $(-\infty, \infty)$. Now an easy computation, involving only the elementary fact that $n/2^n \to 0$ as $n\to\infty$, shows f to be differentiable at 0, with derivative 1. Finally, a glance at the graph of f (Figure 1) together with the reflection property for the graph of the inverse function f^{-1} shows that f^{-1} is not continuous at 0. (Incidentally, f and f^{-1} are both differentiable, with derivative 1, at each $x\notin A\cup B\cup\{1, 0\}$ and neither f nor f^{-1} is continuous at any $x\in A\cup B\cup\{1\}$.)

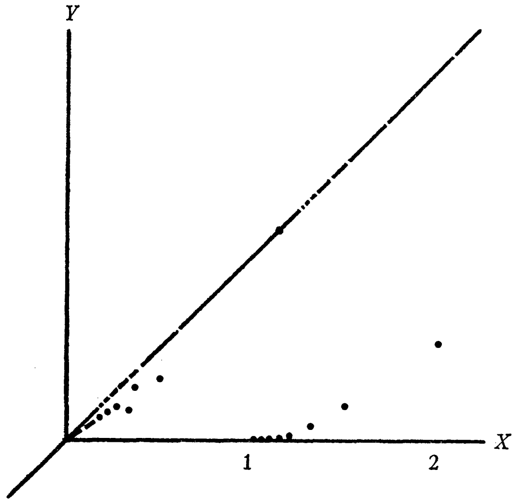

FIG. 1.

It should be noted that while the discontinuities of f are sparse—they form a bounded, countable set having only two limit points—the behavior of f^{-1} near $f(0)$ is due to the fact that 0 is one of these two limit points. Indeed, suppose g is continuous and one-to-one on a neighborhood (a, b) of some point x_0. It is easy to show, using the intermediate value theorem for continuous functions, that g is strictly monotone on (a, b). (Cf. Theorem 2.8.2, p. 167 of [1].) If, moreover, g has a nonzero derivative at x_0, then g^{-1} is differentiable at $g(x_0)$, with derivative $1/g'(x_0)$ (Theorem 3.5.1, p. 198 of [1]).

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AN EXTENSION OF MORLEY'S THEOREM

W. R. SPICKERMAN, East Carolina University

The purpose of this paper is to establish a theorem analogous to Morley's theorem for the exterior angles of a triangle.

THEOREM. *The adjacent trisectors of the exterior angles of a triangle intersect in the vertices of an equilateral triangle.*

The proof follows closely the recent proof of Morley's theorem given by Neidhardt and Milenkovic [1]. The notation used in the proof is given in Figure 1.

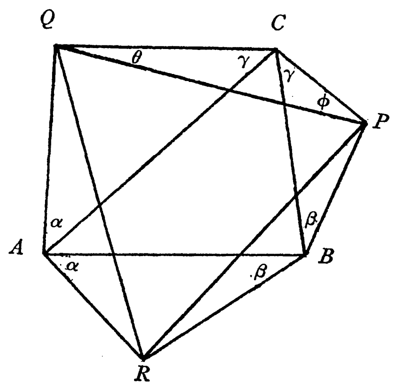

FIG. 1.

Since α, β and γ are $\frac{1}{3}$ of the supplements of $\angle A$, $\angle B$, and $\angle C$, respectively, we have:

$$\alpha+\beta+\gamma = \tfrac{1}{3}(180-\angle A)+\tfrac{1}{3}(180-\angle B)+\tfrac{1}{3}(180-\angle C) = 180-\tfrac{1}{3}(\angle A+\angle B+\angle C) = 120. \tag{1}$$

Furthermore, each angle α, β, and γ is positive and less than 60.

Using (1) to simplify the angle sums in triangles BPC, ACQ, and ARB:

$$\angle BPC = 60+\alpha,\ \angle CQA = 60+\beta,\ \angle ARB = 60+\gamma, \tag{2}$$

and in triangle PCQ:

$$\begin{aligned} \angle PCQ+\theta+\phi &= 180 \\ \theta+\phi &= 120-(\alpha+\beta) \end{aligned} \tag{3}$$

By the law of sines in triangle ABC,

$$BC/\sin\angle A = AC/\sin\angle B = D.$$

But,

$$\sin\angle A = \sin(180-\angle A) = \sin\frac{3(180-\angle A)}{3} = \sin 3\alpha,$$

and

$$\sin \angle B = \sin(180 - \angle B) = \sin \frac{3(180 - \angle B)}{3} = \sin 3\beta.$$

Hence,

$$BC/\sin 3\alpha = AC/\sin 3\beta = D. \tag{4}$$

Also, by the law of sines for triangles BCP, ACQ, and CPQ, respectively,

$$\begin{aligned} PC &= BC \sin \beta/\sin \angle BPC \\ QC &= AC \sin \alpha/\sin \angle AQC \end{aligned} \tag{5}$$

$$\frac{\sin \theta}{\sin \phi} = \frac{PC}{QC}\cdot \tag{6}$$

Substitution into (4) from (3), (2) and the use of the identity

$$\sin 3\omega = 4 \sin \omega(\sin 60 + \omega) \sin(60 - \omega), \text{ yields}$$

$$\begin{aligned} PC &= 4D \sin \beta \sin(60 - \alpha) \sin \alpha \\ QC &= 4D \sin \beta \sin(60 - \beta) \sin \alpha. \end{aligned} \tag{7}$$

Substitution from (7) into (6) gives

$$\frac{\sin \theta}{\sin \phi} = \frac{\sin(60 - \alpha)}{\sin(60 - \beta)}\cdot \tag{8}$$

Finally, from (8) and (3), we have

$$\theta = 60 - \alpha, \quad \text{and} \quad \phi = 60 - \beta. \tag{9}$$

Triangle PCQ is unique because two sides and the included angle are known.

By a similar analysis of appropriate triangles we arrive at the following equations:

$$\begin{array}{lll} \angle RPB = 60 - \gamma & \angle BPC = 60 + \alpha & \angle CPQ = 60 - \beta \\ \angle PQC = 60 - \alpha & \angle CQA = 60 + \beta & \angle AQR = 60 - \gamma \\ \angle QRA = 60 - \beta & \angle ARB = 60 + \gamma & \angle BRP = 60 - \alpha. \end{array} \tag{10}$$

Now, in triangle AQC,

$$\angle CQP + \angle PQR + \angle AQR = 180 - (\alpha + \gamma)$$

$$\angle PQR = 180 - \alpha - \gamma - 60 + \alpha - 60 + \gamma = 60.$$

Similarly, $\angle PRQ = 60$ and $\angle RPQ = 60$.

Thus, triangle PQR is an equilateral triangle.

Reference

1. G. L. Neidhardt and V. Milenkovic, Morley's triangle, this MAGAZINE, 42 (1969) 87–88.

THE GUNPORT PROBLEM

BILL SANDS, University of Manitoba (student)

If one places 2×1 dominoes on an $m \times n$ board (each domino covering exactly two unit squares of the board) until no more dominoes can be accommodated, one notices that there may be a number of 1×1 squares, or "holes", left vacant.

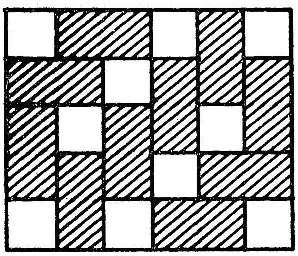

Fig. A

In the example (Figure A), we have placed 10 dominoes on a 5×6 board, and have 10 holes left. A natural question would be to find some bound on the number of holes that may possibly remain. As there is an obvious practical application of this question to the building of castles, the author has termed it the "gunport problem". We will prove the following:

Statement A. For n and m both greater than one, the number of holes is less than or equal to the number of dominoes.

We will first dispose of two special cases.

(1) Suppose $m = 1$. Then it is easily seen that for $n \equiv 1 \pmod 3$, statement A may not be true, whereas for $n \equiv 0$ or $2 \pmod 3$, statement A holds. The case $n \equiv 1 \pmod 3$ fails because one can alternate holes and dominoes with a hole at each end of the $1 \times n$ board, and thus have one more hole than there are dominoes.

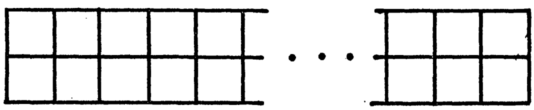

FIG. B

(2) Suppose $m = 2$ (Figure B). One can show that statement A holds for all n in the following way: Consider all vertical dominoes. They will divide the board into smaller $2 \times t_i$ boards ($t_i \leqq n$) which may be considered separately. Thus we need only prove the statement for boards in which the dominoes are all horizontal. Considering the holes and dominoes as forming a closed loop as in Figure C,

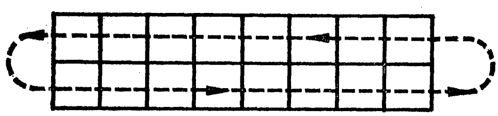

FIG. C

we see that at best, the holes alternate with the dominoes, hence the number of holes is less than or equal to the number of dominoes.

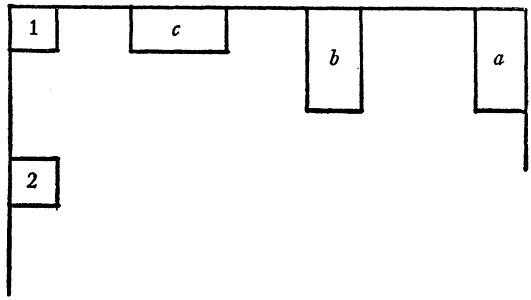

FIG. D

So assume $n \geqq 3$, $m \geqq 3$.

We label the following types of holes and dominoes (refer to Figure D):

holes: type 1—in corner.

type 2—on edge, not of type 1.

dominoes: type a—in corner.

type b—short side on edge, not of type a.

type c—long side on edge, not of type a.

We designate the number of holes of type i as h_i, $i=1, 2$, and the number of dominoes of type j as d_j, $j=a, b, c$. Let the total number of holes be h, and the total number of dominoes be d. Then the number of interior holes $=h-h_1-h_2$, and the number of interior dominoes $=d-d_a-d_b-d_c$. We want to prove $h \leqq d$. To do this we will count the number of distinct pairs (H, D) where H is a hole and D is a domino touching H along one of its edges.

The following are evident from the definitions:

(a) Each hole of type 1 has 2 dominoes touching it.
(b) Each hole of type 2 has 3 dominoes touching it.
(c) Each interior hole has 4 dominoes touching it.

So the number of (H, D) pairs is exactly

$$4(h - h_1 - h_2) + 3h_2 + 2h_1 = 4h - 2h_1 - h_2.$$

Also,

(a) Each domino of type a has at most 2 holes touching it.
(b) Each domino of type b has at most 3 holes touching it.
(c) Each domino of type c has at most 3 holes touching it.
(d) Each interior domino has at most 4 holes touching it.

So the maximum possible number of (H, D) pairs is

$$4(d - d_a - d_b - d_c) + 3d_c + 3d_b + 2d_a = 4d - 2d_a - d_b - d_c.$$

Thus we have

$$4h - 2h_1 - h_2 \leqq 4d - 2d_a - d_b - d_c$$

or

$$(1) \qquad 4h + 2d_a + d_b + d_c \leqq 4d + 2h_1 + h_2.$$

Since at best, the holes along the edge of the board alternate with the dominoes there, we know that the number of holes on the edge does not exceed the number of dominoes on the edge. That is,

$$h_1 + h_2 \leqq d_a + d_b + d_c.$$

So from (1) we have

$$(2) \qquad 4h + 2d_a + d_b + d_c \leqq 4d + h_1 + d_a + d_b + d_c$$

or

$$4h + d_a \leqq 4d + h_1.$$

Now it is obvious that $d_a + h_1 = 4$, so

$$4h + d_a \leqq 4d + 4 - d_a,$$

or

$$4h + 2d_a \leqq 4d + 4,$$

or

$$2h + d_a \leqq 2d + 2.$$

So if $d_a = 2$, 3, or 4, we obtain $h \leqq d$.

Also if $d_a = 1$, then $h \leqq d$, since h and d are integers. So assume $d_a = 0$; that is, every corner has a hole in it. Then

$$2h \leqq 2d + 2, \quad \text{or} \quad h \leqq d + 1.$$

So $h \leqq d$ unless $h = d + 1$. But if we assume $h = d + 1$, then all inequalities above become equalities. Hence, each domino has the maximum number of holes touching it, because in particular (2) and thus (1) become equalities. Consider a corner of the board. (By assumption, there is a hole in every corner.) There are two ways to place dominoes around the hole in this corner, depicted in Figure E. The reader can check that our assumption that each domino has the maxi-

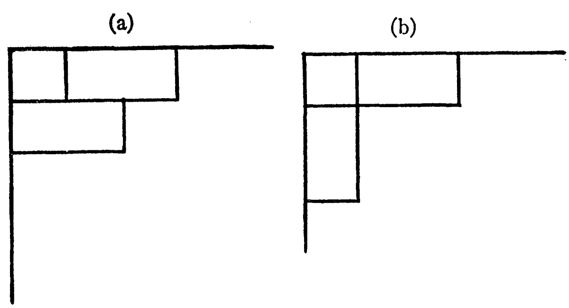

FIG. E

mum number of holes touching it leads to the following repeating patterns (Figure F) arising from (a) and (b) of Figure E:

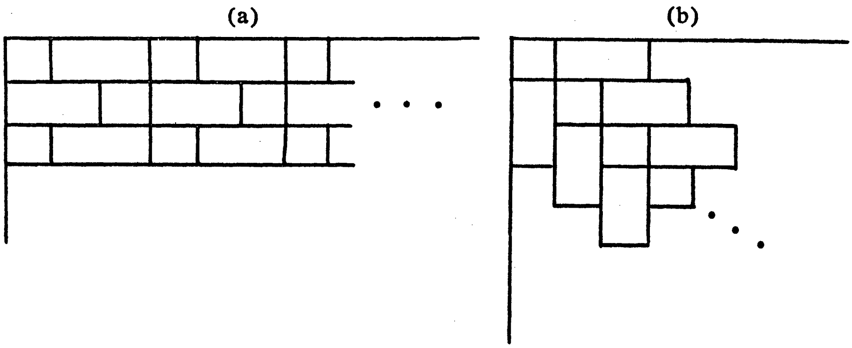

FIG. F

In case F(a), there can be no hole in the right corner.

In case F(b), no rectangular board can be filled with this pattern.

Thus $h \leqq d$ for all values of d_a, and statement A is proven.

Note. If n or $m \equiv 0 \pmod 3$, then the number of holes may be equal to the number of dominoes, as is clear from an inspection of F(a) above. Otherwise, of course, the number of holes will always be less than the number of dominoes. One can find examples to show that if $\min(m, n) \leqq 7$, or if m and n are both $\leqq 9$, the maximum possible number of holes can be obtained. The smallest board for which the author has not been able to find such a maximal case is the 8×10 board, which has been proven to contain a maximum of 26 holes (with 27 dominoes); however the best example so far found has had only 24 holes with 28 dominoes.

ANOTHER DEFINITION OF INDEPENDENCE

BARTHEL W. HUFF, The University of Georgia

The two commonly encountered definitions for the independence of events state that A and B are independent if and only if

$$\text{(i)} \qquad P(AB) = P(A)P(B)$$

or (we shall ignore null and sure events)

$$\text{(ii)} \qquad P(A/B) = P(A).$$

The following alternate definition would seem to be a useful complement to (i) since it only requires conditional probabilities.

DEFINITION. *A and B are independent events if and only if*

$$\text{(I)} \qquad P(A/B) = P(A/B').$$

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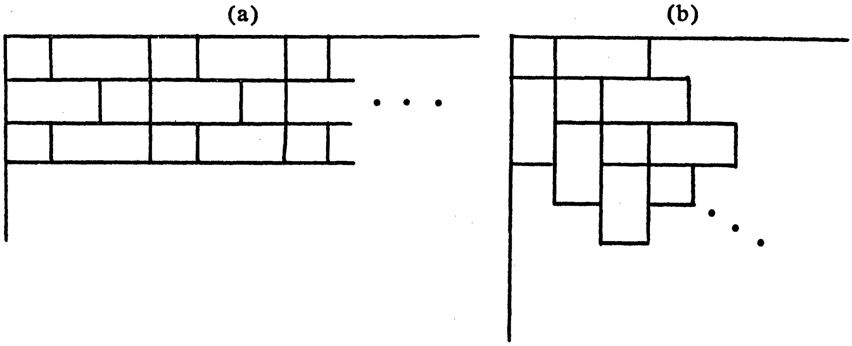

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DEFINITION. *A and B are independent events if and only if*

$$P(A/B) = P(A/B'). \tag{I}$$

(This seems to be a natural definition since it would indicate that the occurrence of A is unaffected by whether or not B occurs.) It is easily seen that (I) and (ii) are equivalent. If A and B are independent, then A and B' are also independent and $P(A/B)=P(A)=P(A/B')$. Conversely, if (I) holds,

$$\begin{aligned} P(A) &= P(A/B)P(B) + P(A/B')P(B') \\ &= P(A/B)P(B) + P(A/B)P(B') \\ &= P(A/B)[P(B) + P(B')] = P(A/B). \end{aligned}$$

Obviously (I) would be useful in situations where conditional probabilities are more easily obtained than probabilities.

Example. A box contains n_1 tags labeled "1" and n_2 tags labeled "2." Urn number 1 contains r_1 red balls and b_1 black balls while urn number 2 contains r_2 red balls and b_2 black balls. A tag is selected at random and then a ball is selected at random from the corresponding urn. Let R denote the event of selecting a red ball and H_1 the event of selecting a tag labeled "1." Since

$$P(R/H_1) = \frac{r_1}{r_1 + b_1} \quad \text{and} \quad P(R/H_1') = \frac{r_2}{r_2 + b_2}$$

we conclude that R and H_1 are independent if and only if $r_1/(r_1+b_1)=r_2/(r_2+b_2)$ and no choice of n_1 and n_2 will compensate for different mixtures in the two urns. If we are restricted to definitions (i) and (ii), a little algebra is required to obtain the same result.

OPTIMAL STRATEGY FOR SERVING IN TENNIS

DAVID GALE, University of California, Berkeley

Some of the things which people habitually do as matters of "common sense" have in recent times become subjects of interesting mathematical analysis. The example *par excellence* is the theory of bluffing in poker as developed by Borel and von Neumann. On a much more elementary level we shall here examine the traditional practice of virtually all tennis players of making their first serve harder than their second. A better description is that the first serve is more "risky," that is, has a lower probability of landing in the court. This would include things like aiming for the sidelines as well as using more force. The question arises as to whether this is a sensible practice or whether it would be better to use the risky serve twice, or the safer twice or even conceivably a safe serve and then a risky serve. The analysis of this question will be on the level of a first course in probability theory and might be useful as illustrative material for such a course. I expect though that it is more illuminating as an example of the formulation and analysis of a simple mathematical model representing a "real life" situation.

Our model will be a simple one indeed. A player has at his disposal a certain

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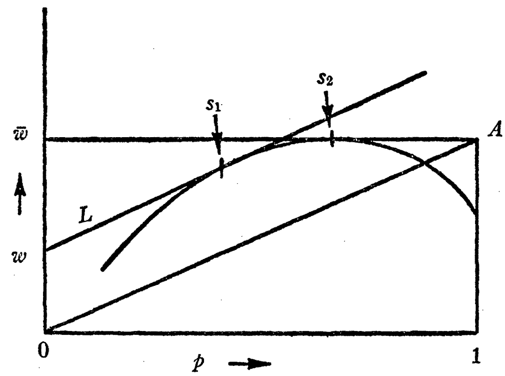
Our model will be a simple one indeed. A player has at his disposal a certain set S, finite or infinite, of possible serves. With each serve s in S are associated two probabilities, $p(s)$ and $q(s)$; $p(s)$ is the probability that s is good (lands in the service area) and $q(s)$ is the conditional probability that if the serve is good it wins the point. The product $p(s)q(s)$ is the probability of winning the point on serve s and will be denoted by $w(s)$. Presumably a serve with a larger p has a smaller q. Intuitively this means the less risky the serve, the easier it is to return. Obviously a serve which is both more risky and easier to return than another should never be used.

A *strategy* consists of an ordered pair (s_1, s_2) from $S\times S$ where s_1 and s_2 are the chosen first and second serves. The payoff $P(s_1, s_2)$ is the probability of winning the point when these serves are used. The reader will easily see that

$$P(s_1, s_2) = w(s_1) + (1 - p(s_1))w(s_2), \tag{1}$$

for the point will be won either on the first or second serve (but not on both) and the respective probabilities are the two terms on the right of (1). An optimal strategy is one which maximizes (1) and it is easy to see how to find such a strategy. Clearly s_2 must be chosen to maximize $w(s)$. Letting $\bar{w}$ be the value of this maximum, one then chooses s_1 to maximize $w(s)-\bar{w}p(s)$. The figure below gives a simple graphical procedure. For each s, plot the points $(p(s), w(s))$.



The figure is drawn as though the player had a continuum of possible serves. The serve s_2 corresponds to the highest point on the graph. The line OA has slope $\bar{w}$ and line L is parallel to OA and "just touching" (supporting) the graph at the point corresponding to s_1.

What does this tell us? For the case shown in the figure one does indeed use two different serves, the first being more risky. However, it is perfectly possible to have cases where $s_1=s_2$, namely if the line L happened to support the graph at s_2. It is clear, however, that it would never make sense to serve a more risky serve as second serve.

Finally, consider the simplest case of a player with only two serves s_1 and s_2 in his arsenal with coordinates (p_1, w_1) and (p_2, w_2) where $p_1<p_2$. Then we see from (1) that (s_1, s_2) is an optimal strategy if and only if

$$w_2 \geqq w_1 \geqq w_2(1 - (p_2 - p_1)). \tag{2}$$

If the first inequality fails, then the player should use his risky serve both times while if the second fails he should use two safe serves.

The "common sense" of the usual practice is now easy to describe. Why should a person be willing to take a greater risk on his first than on his second serve? Clearly because the penalty for failure on the second serve is greater. More formally, on the second serve the expected payoff for a good serve is q_2 and the payoff for a bad serve is zero so one must maximize $p_2q+(1-p_2)0=w_2$. On the first serve the payoff for a good serve is q_1 but the payoff for a bad serve is the opportunity to make a second "gamble" with expected payoff w_2, so on the first serve one maximizes $p_1q_1+(1-p_1)w_2$, which is formula (1) again.

A NOTE ON THE *N*TH TERM OF THE FIBONACCI SEQUENCE

ERWIN JUST, Bronx Community College, City University of New York

The explicit expression for the nth term of the Fibonacci sequence defined by $F_n=F_{n-1}+F_{n-2}$, $F_1=F_2=1$, may be quickly obtained by the method described in this note.

It is easily proved by induction that if $x^2=x+1$, then $x^n=F_nx+F_{n-1}$, for $n\geqq 2$. Thus, if α and β denote the roots of $x^2=x+1$, then $\alpha^n=F_n\alpha+F_{n-1}$ and $\beta^n=F_n\beta+F_{n-1}$. By subtraction we obtain $\alpha^n-\beta^n=F_n(\alpha-\beta)$ or, alternatively, $F_n=(\alpha^n-\beta^n)/(\alpha-\beta)$. Since $\alpha=(1+\sqrt{5})/2$ and $\beta=(1-\sqrt{5})/2$ are the roots of $x^2=x+1$, it follows that $F_n=[(1+\sqrt{5})^n-(1-\sqrt{5})^n]/2^n\sqrt{5}$.

References

1. G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Oxford, London, 1960, p. 148.
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ON THE DENSEST PACKING OF EQUAL SPHERES IN A CUBE

MICHAEL GOLDBERG, Washington, D. C.

1. Introduction. Many varieties of packing problems have been studied. The best summary of these investigations is found in the book by Fejes Tóth [1]. One of the most interesting is the problem of the determination of the densest packing of spheres. Infinite packing and packing within special shapes are treated by Boerdijk [2]. The densest packing of equal spheres within a cube has been determined by Schaer [3] up to nine spheres. The present paper describes arrangements up to twenty-seven spheres, and several sequences of arrangements for numbers of spheres beyond twenty-seven. All of these are suggested merely as conjectures of the best arrangements. It is hoped that someone may prove that some of them are the best, or find some better arrangements.

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In earlier papers [6, 7], the author studied the packing of equal circular rings on the surface of a sphere, and the packing of equal circles in a square, and found that the investigation of the various possible axially symmetric packings usually yielded the most efficient packings. The method seems to be equally effective in the study of the packing of equal spheres in a cube. There are three different axes of symmetry in a cube, namely, the join of opposite vertices, the join of the midpoints of opposite edges, or the join of the midpoints of opposite faces. These give two-fold, three-fold or four-fold symmetry about the axis. Each of these should be tried.

2. Notation. Let the spheres be arranged in layers which are normal to an axis. Then, a notation for this arrangement consists of the total number of spheres and the numbers in each of the successive layers. For example, the notation $7\{3,1,3\}$ means that seven spheres are arranged in three layers, that three spheres are equally spaced in the first and third layers, and that a solitary sphere is in the middle layer. The notation $14\{(1, 4),4,(1,4)\}$ means that there are three layers, that in the first and last layers there is a sphere surrounded by four equally-spaced spheres, and that four equally-spaced spheres are in the middle layers.

Take a cube inside of the given cube, with faces parallel to the given cube, and at a distance equal to the radius of the given spheres. Following the practice of Schaer, we let the edge of this smaller cube be unity, and we designate the diameter of the given spheres by m_n, where n is the number of equal spheres to be packed.

The density of packing $d(n)$ is the total volume of the n spheres divided by the volume of the cube within which they are packed. Hence, $d(n) = \pi n m^3/6(1+m)^3$. If, instead, the spheres are taken as having unit diameter, then the edge of the cube within which they are packed is designated by $e(n)$. Hence, $e(n) = (1+m)/m = 1+1/m$, and $d(n) = \pi n/6e^3$.

3. Several new arrangements. The following arrangements were investigated; $10\{4,1,1,4\}$, $10\{3,1,3,3\}$ and $10\{1,4,2,2,1\}$. The first, shown in Figure 1, yielded the value $m_{10} = (\sqrt{10}-1)/3 = 0.721$. The others give smaller values.

The arrangement $11\{3,1,3,1,3\}$ was investigated. It gave the value $m_{11} = \sqrt{2}/2 = 0.7071$. But this arrangement allows the addition of three more spheres in vacant spaces and becomes the arrangement $14\{3,1,6,1,3\}$.

If three spheres are omitted from the arrangement of 14 spheres shown in Figure 2a, then the top spheres can be moved toward each other until they touch, as shown in Figure 2b. Since the enclosing cube no longer touches the top spheres, all the spheres may be enlarged. In this process, the middle sphere at the bottom, shown in Figure 2c, no longer touches the bottom, and the top spheres are raised until they touch the top of the enclosing cube. For this arrangement, $m_{11} = 0.7077$.

The arrangement of 14 spheres, described in the foregoing, can also be described by $14\{(1,4),4,(1,4)\}$ and is shown in Figure 3. This is the arrangement found everywhere in normal piling, or spherical close-packing, of an infinite set of spheres. There are various modifications of this infinite packing, all of the

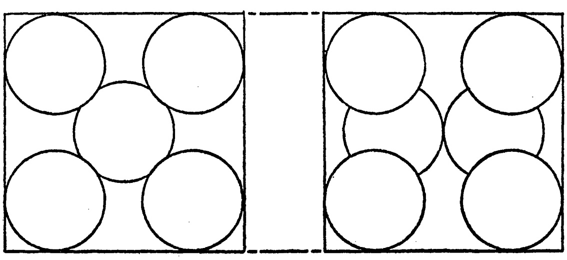

FIG. 1. 10{4, 1, 1, 4}

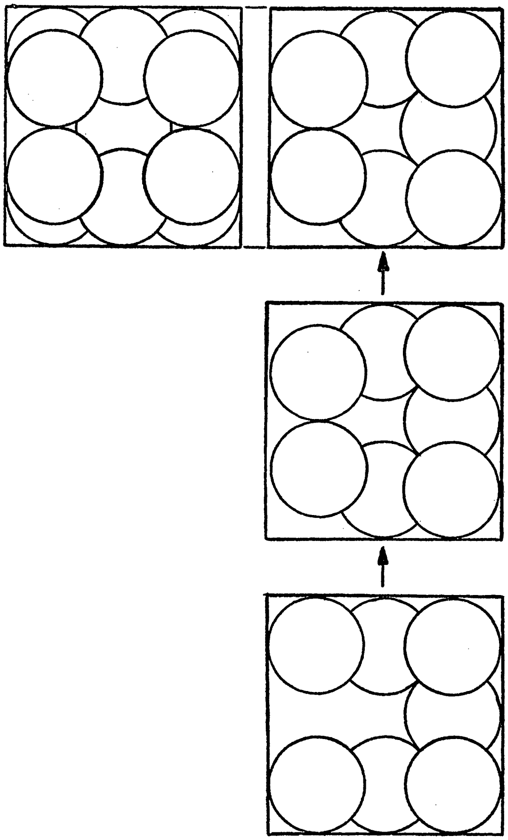

FIG. 2. 11{(2, 2), 2, 1, 4}

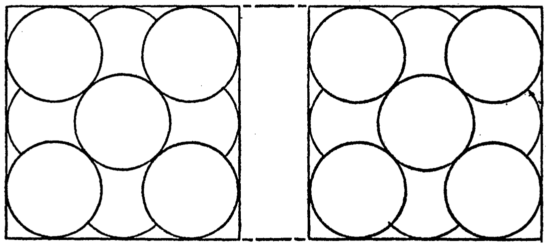

FIG. 3. 14{(1, 4), 4, (1, 4)}

same density. No denser packing is known, but the proof of the nonexistence of a denser packing has eluded mathematicians. Hence, although the arrangement of 14 spheres in a cube, shown in Figure 3, is obviously the densest, a rigorous proof is still not available.

It seems likely that the best arrangements for 13 spheres and 12 spheres may be obtained by the omission of the appropriate number of spheres from the arrangement 14{(1,4),4,(1,4)}. No better arrangements have been found.

4. Packing of 15 spheres. The various possibilities of packing and the different densities obtained thereby are well illustrated in the case of 15 spheres. The arrangement 15{(1,4),(1,4),(1,4)} consists of three layers, each of five spheres. The middle layer must be shifted in the direction of a face diagonal to make a cube. Then $m=0.570$ and $d(n)=0.377$, $n/e^3=0.720$.

In another arrangement, six spheres are placed in the bottom layer, and a similar set of six spheres is placed in the top layer. This leaves space for three spheres in the middle layer. Since the layers are loose in the cube, we can improve the packing by compressing the sides of the top and bottom layers so that some of the spheres are pushed out of the plane. In Figure 4, ($m=0.589$, $d(n)=0.401$, $n/e^3=0.766$), the two corner spheres are kept in the plane while the other four spheres are pushed out of the plane. When this is optimized, 14 of the spheres are locked in place while the center sphere is loose.

In further modifications, shown in Figures 6 and 7, ($m=0.598$, $d(n)=0.412$, $n/e^3=0.787$), the arrangements are staggered. Then, 14 of the spheres are locked in place while the center sphere is loose.

In another modification, shown in Figure 5, ($m=0.595$, $d(n)=0.409$, $n/e^3=0.781$), the two corner spheres are pushed out of the plane while the other four spheres are kept in the plane. When this is optimized, 12 of the spheres are locked in place, while the three spheres in the middle layer are loose.

In each of the foregoing cases, the value of m is less than the value $m=0.601$ which is attainable for $n=18$. Hence, the arrangement which gives the largest known value of m for $n=15$ is obtained by omitting three spheres from the arrangement of 18 spheres described in the next section.

5. Packing of 18 spheres. An efficient packing of 18 spheres in a cube, shown in Figure 8, is represented by 18{(1,4),4,(1,4),4} or its equivalent 18{(2,1,2,1), (1,2,1,2),(2,1,2,1)}. In the front view and in the plan view of Figure 8, there are six spheres touching the face of the cube. These six spheres are represented as six circles in a square. This is the optimum arrangement of six circles in a square as shown by Schaer [5]. The 18 spheres are made of three such layers, the middle layer being reversed left-to-right. This layering is the same in the top view as in the front view. In the left and right side views, the layering is (1,4),4,(1,4),4 or 4,(1,4),4,(1,4). Here $m_{18}=0.601$, $e=2.6641$, and $n/e^3=0.992$.

Attempts at finding packings of 16 and 17 spheres did not produce efficient arrangements. For example: for 16{2,(1,2,2),2,(1,2,2),2}, $m=0.595$; for 17{(1,6),3,(1,6)}, $m=0.52$; for 17{1,3,3,3,3,3,1}, $m=0.548$. Since these values of m are less than $m_{18}=0.601$, it is likely that the best values for 17 spheres and 16 spheres are obtained by omitting one or two spheres from 18{(1,4),4,(1,4),4}.

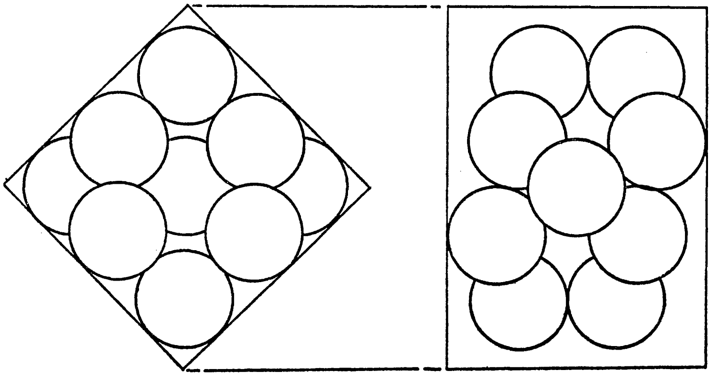

FIG. 7. 15{2, 4, (1, 2), 4, 2}

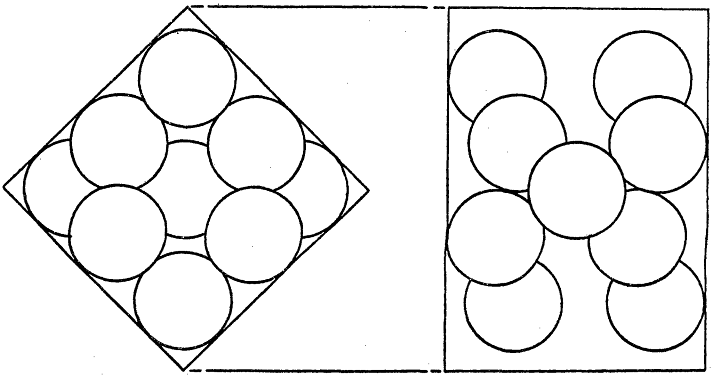

FIG. 6. 15{(1, 2), (1, 2), (1, 2), (1, 2), (1, 2), (1, 2)

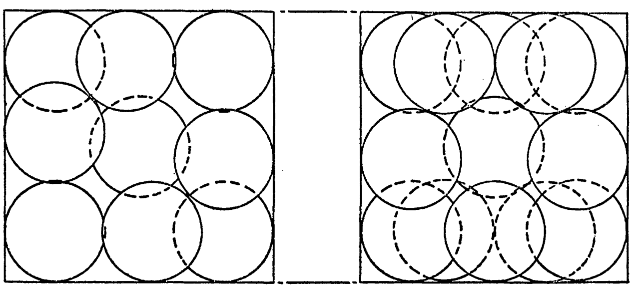

FIG. 5. 15{(4), 2, (1, 2), 2, (4)}

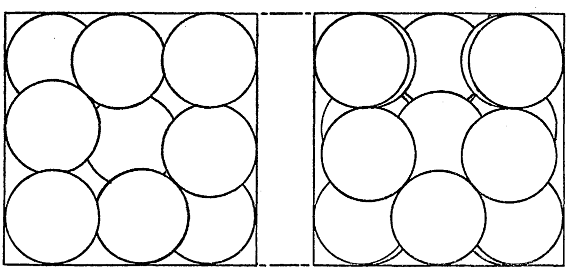

FIG. 4. 15{2, (4), 1, 2, (4), 2}

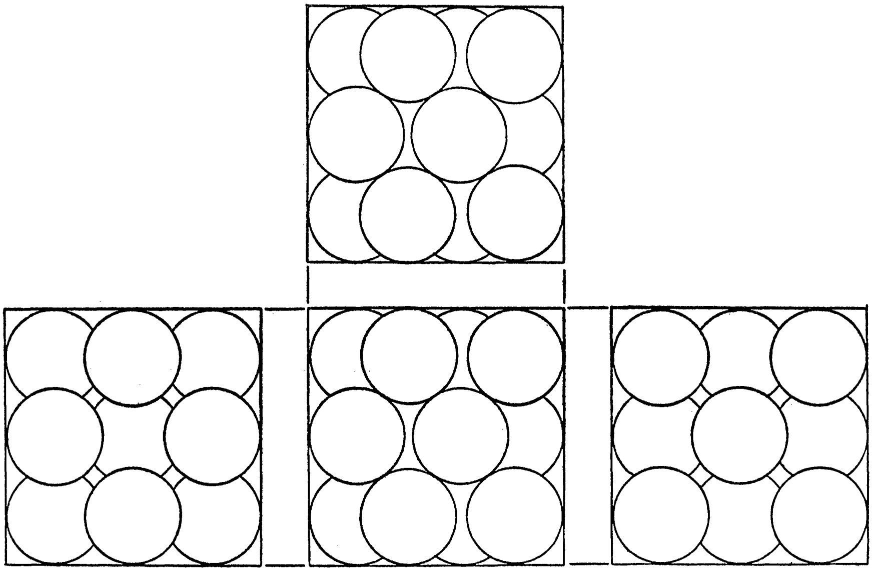

FIG. 8. 18{(1, 4), 4, (1, 4), 4}

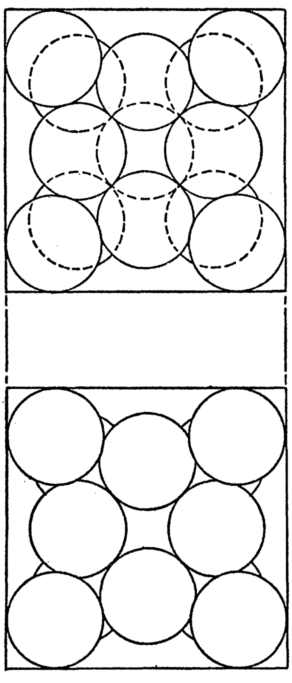

FIG. 9. 21{4, 4, (1, 4), 4, 4}

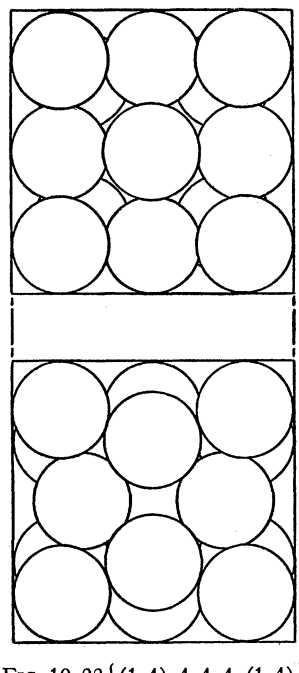

FIG. 10. 22{(1, 4), 4, 4, 4, (1, 4)}

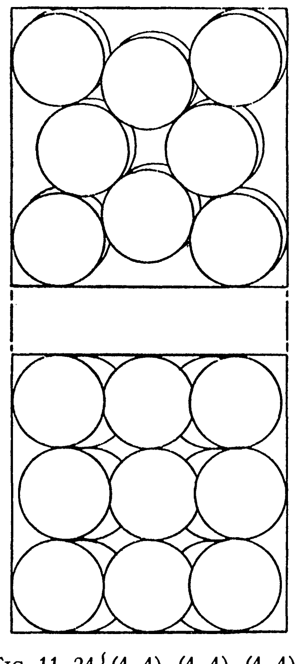

FIG. 11. 24{(4, 4), (4, 4), (4, 4)}

6. Packing of 21 spheres. In the packing of 14 spheres, shown in Figure 3, there is a layer of 5 spheres, then a layer of 4 spheres, and finally another layer of 5 spheres. If each layer of 5 spheres is replaced by a layer of 4, and the layer of 4 spheres in the middle is replaced by a layer of 5, we obtain a cluster of 13 spheres which pack in the same cube as did the 14 spheres. At the vertices of the cube, there are large spaces which are not quite large enough for another sphere within the cube. However, if spheres are placed in these eight locations, we obtain a cluster of 21 spheres which can be packed efficiently into a larger cube. The arrangement is shown in Figure 9, where the views of all the faces are the same. The notation for this arrangement is $21\{4,4,(1,4),4,4\}$. In this case, $m_{21}=3\sqrt{2}/8=0.5303$ and $e=2.8856$, $n/e^3=0.8740$. Note that the eight spheres, which are seen in each view, are nearly in a plane and that they approximate the optimum arrangement of eight circles in a square [4, 5]. For this reason, it is expected that this arrangement of 21 spheres should be the best.

Here, again, it is likely that the best arrangement for 20 spheres and 19 spheres may be obtained by omitting one or two spheres from this arrangement of 21 spheres.

7. Packing of 22 and 24 spheres. The packing of 21 spheres suggests another arrangement of (1,4)-layers and 4-layers to obtain the packing $22\{(1,4),4,4,4,(1,4)\}$ as shown in Figure 10. This yields $m=(\sqrt{6}-\sqrt{2})/2=0.5176$, $e=2.9318$ and $n/e^3=0.8730$.

The use of the efficient (4,4)-layers yields $24\{(4,4),(4,4),(4,4)\}$ as shown in Figure 11. In this arrangement, the middle layer is slightly offset from the top and bottom layers by a shift along the diagonal of the square. This yields $m=\{2(\sqrt{2}+\sqrt{6})-\sqrt{5-2\sqrt{3}}\}/2(3+2\sqrt{3})=0.5019$, $e=2.9924$ and $n/e^3=0.8954$.

8. Limiting values. As the number of spheres increases, there are special numbers for which very efficient packings are possible. These are the numbers of spheres in portions of the infinite regular cubic close-packing. In one such sequence, there are exactly p regularly-spaced spheres along the edge of the cube. The number of spheres in an outer layer is $p^2+(p-1)^2$; and there are p such layers. The number of spheres in an adjacent layer is $p^2+(p-1)^2-1$; and there are $(p-1)$ such layers. Hence, the total number of spheres in the cube is given by

$$n = p[p^2 + (p-1)^2] + (p-1)[p^2 + (p-1)^2 - 1] = p(4p^2 - 6p + 3).$$

Then, $m_n=\sqrt{2}/(2p-2)$ and the length of the edge e is given by $e=1+1/m=1+\sqrt{2}(p-1)$. Also, $d(n)=(\pi/6)(n/e^3)$. A sequence of values is given in the following table:

p	2	3	4	5	6
n	14	63	172	365	666
e	2.4142	3.8284	5.2426	4.6568	8.0710
n/e^3	0.994	1.123	1.194	1.237	1.267

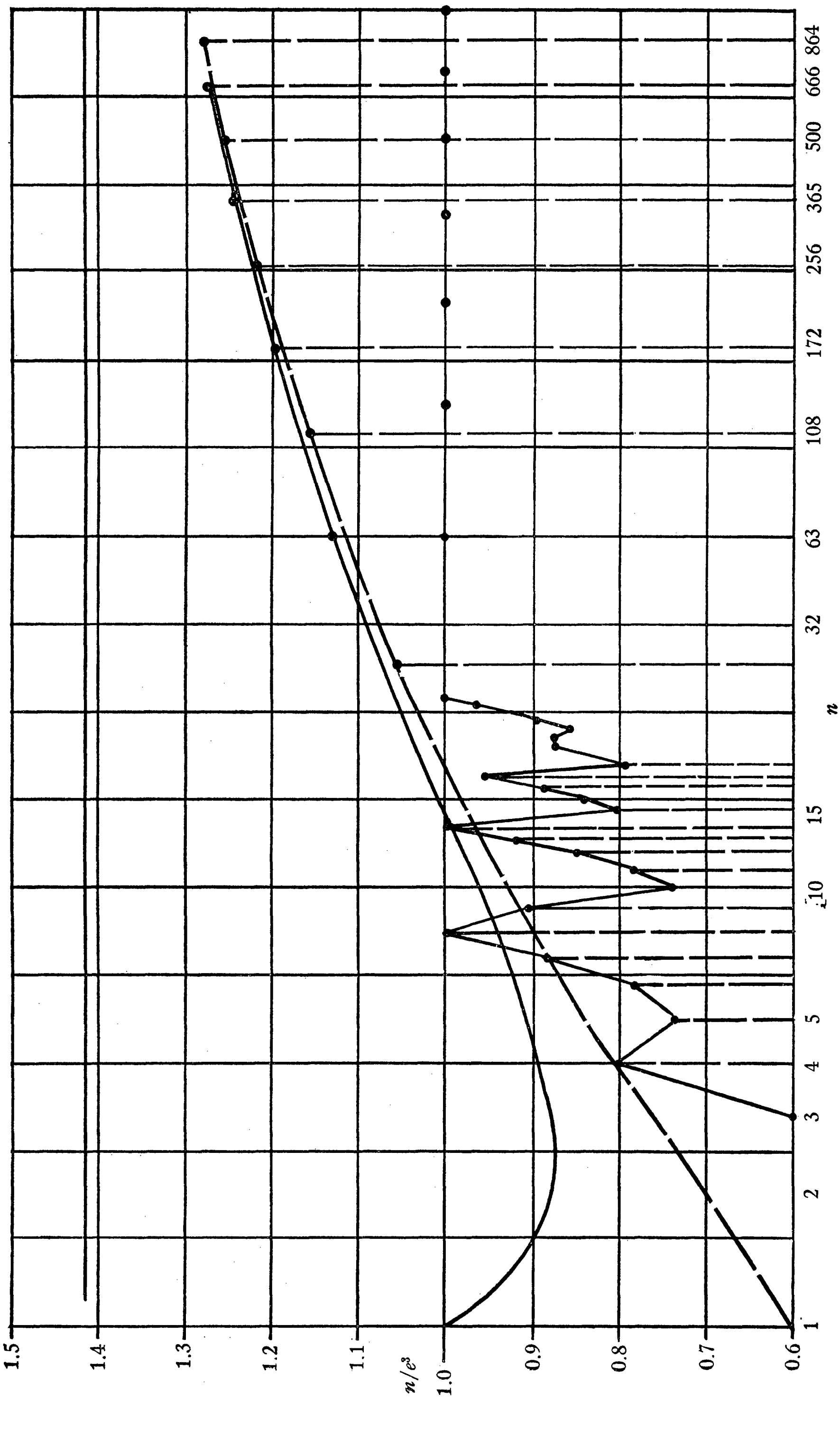

FIG. 12. Graph of n/e^3

These values are represented on Figure 12 by the circled points on the *solid* curve which is the graph of the function n/e^3.

If the cube has one layer less, and the number of layers is $2r$, then the total number of spheres is given by $n=r^3/2$. Then,

$$m_n = \sqrt{2}/(r-1),\ e = 1+\sqrt{2}(r-1)/2, \quad \text{and} \quad d(n) = (\pi/6)(n/e^3).$$

A sequence of values is given in the following table:

r	2	4	6	8	10	12
n	4	32	108	256	500	864
e	1.7071	3.1213	4.5355	5.9497	7.3639	8.7781
n/e^3	0.804	1.052	1.158	1.216	1.252	1.277

These values are represented on Figure 12 by the circled points on the *dashed* curve which is a graph of the function n/e^3.

For $n \leqq 27$, the circled points on Figure 12 represent the best values obtained. The data for these cases are summarized in the following table.

n	Arrangement	m	e	n/e^3
1	1		1.000	1.000
2	2	1.732	1.577	.510
3	3	1.414	1.707	.603
4	2,2 or 1,3	1.414	1.707	.804
5	1,3,1	1.118	1.894	.737
6	1,4,1 or 3,3	1.031	1.969	.788
7	1,3,3	1.009	1.991	.887
8	4,4 or 1,3,3,1	1.000	2.000	1.000
9	4,1,4	.866	2.154	.900
10	4,1,1,4	.721	2.387	.736
11	(2,2),2,1,4	.7077	2.412	.782
12	Two from 14	.7071	2.414	.852
13	One from 14	.7071	2.414	.924
14	(1,4),4,(1,4) or 1,3,3,3,1	.7071	2.414	.995
15	Three from 18	.601	2.6641	.792
16	Two from 18	.601	2.6641	.846
17	One from 18	.601	2.6641	.899
18	(1,4),4,(1,4),4	.601	2.6641	.952
19	Two from 21	.5303	2.8856	.791
20	One from 21	.5303	2.8856	.832
21	4,4,(1,4),4,4 or 1,3,3,(1,6),3,3,1	.5303	2.8856	.8740
22	(1,4),4,4,4,(1,4)	.5176	2.9318	.8730
23	One from 24	.5019	2.9924	.8584
24	(4,4),(4,4),(4,4)	.5019	2.9924	.8954
25	Two from 27	.5000	3.0000	.925
26	One from 27	.5000	3.0000	.963
27	1,3,3,3(1,6),3,3,3,1	.5000	3.0000	1.000

9. Cubic lattice packing. When the centers of the spheres are the points of a cubic lattice, then $n=p^3$, $e=p$, and $d(n)=(\pi/6)(n/e^3)=\pi/6$. These values are represented on Figure 12 by the sequence of circled points along the line $n/e^3=1$. For $p=2$, $(n=8)$, the best value of n/e^3 is obtained. However, for larger values, the arrangements obtained from the nearby regular close-packing gives better values. It is only for $n=1$, 8 and 27 that cubic lattice packing seems to be the best.

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MODULAR PALINDROMES

RODNEY T. HANSEN, Montana State University

I. Introduction. A word, phrase, or sentence that is the same when read backward or forward is called a *palindrome*. Examples of word palindromes are "mom", "dad", "level", and "radar". In the realm of mathematics, numbers which are unchanged by a reversion of digits are also called palindromes. Numbers such as 1, 22, 303, and 25,252 are examples of numerical palindromes.

In this paper the concept of a modular palindrome is defined, and several general classes of modular palindromes are discussed. A result is given which determines for any positive integer a and base b, all m such that a is a palindrome modulo m. Several methods of constructing these numbers as well as some of their interesting properties are also given.

As is well known, a positive integer a may be expressed uniquely in base $b>1$ in the form

$$a = a_n \times b^n + a_{n-1} \times b^{n-1} + \cdots + a_1 \times b^1 + a_0$$

which written in positional notation is

$$a = (a_n a_{n-1} \cdots a_1 a_0)_b,$$

with $a_n \neq 0$ and integers a_i between 0 and $b-1$ inclusively for all i. Integer a is called a *palindrome modulo m in base b*, written palindrome (mod m, b), if

$$a = (a_n a_{n-1} \cdots a_1 a_0)_b \equiv (a_0 a_1 \cdots a_{n-1} a_n)_b \pmod{m}$$

9. Cubic lattice packing. When the centers of the spheres are the points of a cubic lattice, then $n=p^3$, $e=p$, and $d(n)=(\pi/6)(n/e^3)=\pi/6$. These values are represented on Figure 12 by the sequence of circled points along the line $n/e^3=1$. For $p=2$, $(n=8)$, the best value of n/e^3 is obtained. However, for larger values, the arrangements obtained from the nearby regular close-packing gives better values. It is only for $n=1$, 8 and 27 that cubic lattice packing seems to be the best.

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$$a = (a_n a_{n-1} \cdots a_1 a_0)_b \equiv (a_0 a_1 \cdots a_{n-1} a_n)_b \pmod{m}$$

where m is a fixed positive integer. We will let a^1 denote $(a_0a_1 \cdots a_{n-1}a_n)_b$ which is the integer formed by reversing the digits of a.

Let us consider several examples found in base b equal to 10.

(i) Let $m=4$. The reader may easily check that

$$15 \equiv 51 \pmod 4$$

$$37 \equiv 73 \pmod 4$$

and

$$48 \equiv 84 \pmod 4.$$

(ii) Now let $m=15$. Then

$$16 \equiv 61 \pmod{15}$$

$$38 \equiv 83 \pmod{15}$$

and

$$49 \equiv 94 \pmod{15}.$$

Other examples of palindromes for $m=4$ or $m=15$ may be quickly found.

II. Examples and generalizations. We first note that since $10 \equiv 1 \pmod 9$ $\equiv 1 \pmod 3$, any positive integer is a palindrome (mod m, 10) for m equal to 3 and 9. Also every positive integer with an odd number of digits is a palin*drome (mod 11, 10). The observations are special cases of the following results.* Let integer b be the base of the positional representation.

A. Positive integer a is a palindrome (mod d, b) for any divisor d of $b-1$.

B. Integer $a=(a_na_{n-1} \cdots a_1a_0)$ is a palindrome (mod $b+1$, b) if n is even.

Proof of B. We have

$$a = \sum_{i=0}^{n} a_ib^i \equiv \sum_{i=0}^{n} a_i(-1)^i \equiv \sum_{i=0}^{n} a_i(-1)^{n-i} \equiv a^1 \pmod{b+1}.$$

Thus, in ordinary decimal representation, it follows that

$$31720 \equiv 02713 \pmod{11}, \quad \text{and}$$

$$5730215 \equiv 5120375 \pmod{11}.$$

Note that each of the above numbers is congruent to $7 \pmod{11}$.

III. Basic result. To find all palindrome moduli distinct from divisors of $b-1$ for given integer a, we need only find the canonical representation of the following sum.

THEOREM 1. *Let $a=(a_na_{n-1} \cdots a_1a_0)_b$ be given. Then for all integers $d>0$ such that*

$$d \,\Big|\, \frac{1}{b-1} \sum_{i=0}^{[(n-1)/2]} (b^{n-i} - b^i)(a_{n-i} - a_i)$$

we have $a \equiv a^1 (\text{mod } d)$. ("$[(n-1)/2]$" *denotes the greatest integer not exceeding* $(n-1)/2$.)

Proof. If d is any positive divisor of the given sum, then

$$\sum_{i=0}^{[(n-1)/2]} (b^{n-i} - b^i)(a_{n-i} - a_i) \equiv 0 \pmod{d}.$$

Thus

$$\begin{aligned}\sum_{i=0}^{[(n-1)/2]} b^i(a_{n-i} - a_i) &\equiv \sum_{i=0}^{[(n-1)/2]} (b^{n-i} - b^i)(a_{n-i} - a_i) + \sum_{i=0}^{[(n-1)/2]} b^i(a_{n-i} - a_i) \pmod{d} \\ &\equiv \sum_{i=0}^{[(n-1)/2]} b^{n-i}a_{n-i} - \sum_{i=0}^{[(n-1)/2]} b^{n-i}a_i \pmod{d}.\end{aligned}$$

Therefore, for n odd, we have

$$\begin{aligned}a = \sum_{i=0}^{n} b^i a_i &= \sum_{i=0}^{[(n-1)/2]} b^i a_i + \sum_{i=0}^{[(n-1)/2]} b^{n-i}a_{n-i} \\ &\equiv \sum_{i=0}^{[(n-1)/2]} b^i a_{n-i} + \sum_{i=0}^{[(n-1)/2]} b^{n-i}a_i \pmod{d} \equiv a^1 \pmod{d};\end{aligned}$$

and adding the common term $b^{n/2}\, a_{n/2}$ to the above sum for even n, completes the proof.

For example, consider $a = (6135)_{10} = a_3a_2a_1a_0$, then

$$\frac{1}{9} \sum_{i=0}^{[(3-1)/2]} (10^{3-i} - 10^i)(a_{3-i} - a_i) = 91 \quad \text{and so}$$

$$(6135)_{10} \equiv (5316)_{10} \pmod{d}$$

for any positive divisor d of 91. Similarly

$$(53721)_{10} \equiv (12735)_{10} \pmod{d}$$

for any positive d that divides 4554.

IV. Construction of palindromes from palindromes. Three rather simple ways of making such constructions are first given.

THEOREM 2. *If* $a \equiv a^1 (\text{mod } m, b)$ *and* a *has leading coefficient* $a_n \neq 0$, *then for any integer* k *such that*

$$\max_{0 \leq i \leq n} \{a_i\} \leq k < b,$$

the number

$$a^* = (k - a_n)b^n + (k - a_{n-1})b^{n-1} + \cdots + (k - a_1)b + (k - a_0)$$

is also a palindrome (mod m, b).

Proof. We write

$$a^* = \sum_{i=0}^{n} (k - a_i)b^i = \left(\sum_{i=0}^{n} kb^i \right) - a$$
$$\equiv \left(\sum_{i=0}^{n} kb^{n-i} \right) - a^1 \equiv (a^*)^1 \pmod{m}.$$

A closely related result follows.

THEOREM 3. *If a is a palindrome* (mod m, b) *and* $0 \leqq c \leqq \min \{a_i\}$, *then*

$$d = \sum_{i=0}^{n} (a_i - c)b^i$$

is also a palindrome (mod m, b). (*An ordinary palindrome is any number d written in positional notation in base* 10 *such that* $d = d^1$.)

THEOREM 4. *If* $(a_1a_0)_b \equiv (a_0a_1)_b$ (mod m) *and if* $c = (c_1c_2 \cdots c_s)_b$ *is an ordinary or modular m palindrome, then*

$$d = (a_1c_1c_2 \cdots c_sa_0)_b$$

is also a palindrome (mod m, b).

Proof. Now $(a_1a_0)_b \equiv (a_0a_1)_b \pmod{m}$ implies

$$(b - 1)(a_1 - a_0) \equiv 0 \pmod{m}.$$

Since $(b-1) \mid (b^{s+2}-1)$ for any positive integer s and $b > 1$, we have

$$\left\{\frac{b^{s+2} - 1}{b - 1}\right\} \{(b - 1)(a_1 - a_0)\} \equiv 0 \pmod{m}$$

or

$$b^{s+2} \times a_1 + a_0 \equiv b^{s+2} \times a_0 + a_1 \pmod{m}.$$

We now show the desired congruence:

$$d = a_1 \times b^{s+2} + c_1 \times b^{s+1} + \cdots + c_{s-1} \times b^2 + c_s \times b + a_0$$
$$= a_1 \times b^{s+2} + b(c) + a_0$$
$$\equiv a_0 \times b^{s+2} + b(c^1) + a_1 \pmod{m} \equiv d^1 \pmod{m}.$$

COROLLARY. *If* $(a_1a_0)_b \equiv (a_0a_1)_b$ (mod m), *then*

$$(a_1\overbrace{00 \cdots 0}^{s \text{ terms}}a_0)_b \equiv (a_0\overbrace{00 \cdots 0}^{s \text{ terms}}a_1)_b \pmod{m}$$

for any positive integer s.

V. The reader may use the results of the preceding section to construct modular palindromes. To aid in this endeavor a small table of two digit non-ordinary palindromes (mod m, 10) is given.

modulus m	*nonordinary two digit palindromes* (mod *m*, 10)
4	15, 19, 26, 37, 40, 48
5	16, 27, 38, 43, 49, 50
6	13, 24, 26, 28, 31, 35, 37, 39, 40, 42, 46, 48
7	18, 29
8	19
9	all
10	none
11	none
12	15, 19, 26, 37, 40, 48
13	none
14	none
15	16, 27, 38, 49, 50
16	none
17	none
18	13, 15, 17, 19, 24, 26, 28, 31, 35, 37, 39, 40, 42, 46, 48

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TWO DEFINITIONS OF TANGENT PLANE

CLIFFORD A. LONG, Bowling Green University

Frequently, after a differentiable function of 2 variables has been defined, the notion of a tangent plane to a surface defined by this function is described in terms of an equation using the partial derivatives of the function [**1**, p. 214], as a collection of tangent vectors to the surface at the point [**2**, p. 572], or in terms of a normal vector to the surface [**3**, p. 336], with related properties of the function indicated.

The purpose of this note is to show the equivalence of the following definitions of a tangent plane to a surface defined by a continuous function, without considering differentiability of the function. We let the surface S be defined by $(x, y, f(x, y))$ and restrict ourselves to nonvertical tangent planes.

DEFINITION I. *The tangent plane to* S *at* P_0 *is defined as the plane formed by tangent lines at* P_0 *to curves,* $(x(t), y(t), f(x(t), y(t)))$ *where* x *and* y *are differentiable, if indeed these tangent lines all exist and lie in the same plane* [**4**, p. 50].

DEFINITION II. *The tangent plane to* S *at* P_0 *is defined as the plane* M *for which the angle between line* P_0P*, where* P *is on* S*, and the plane* M *approaches zero as* P *approaches* P_0 [**5**, p. 216].

THEOREM. *Definitions* I *and* II *are equivalent.*

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DEFINITION I. *The tangent plane to S at P_0 is defined as the plane formed by tangent lines at P_0 to curves, $(x(t), y(t), f(x(t), y(t)))$ where x and y are differentiable, if indeed these tangent lines all exist and lie in the same plane* [4, p. 50].

DEFINITION II. *The tangent plane to S at P_0 is defined as the plane M for which the angle between line P_0P, where P is on S, and the plane M approaches zero as P approaches P_0* [5, p. 216].

THEOREM. *Definitions* I *and* II *are equivalent.*

Proof. (I→II) Suppose there exists a tangent plane by Definition I. Without loss of generality, we may assume that this plane is horizontal and that $P_0 = (0, 0, 0)$. Let Γ be a curve on surface S defined by $P(t) = (x(t), y(t), f(x(t), y(t))$ where curve γ defined by $p(t) = (x(t), y(t))$ is at least once differentiable at $(0, 0)$. All such curves Γ have tangent lines which all lie in the plane $z = 0$. (We say that a line is tangent to a curve if and only if the angle between the line and line P_0P approaches zero as P approaches P_0 along the curve.) Let L be the tangent line to Γ and consider angles α, β, θ between pairs of lines L and P_0p, P_0p and P_0P, P_0P and L, respectively, as indicated in Figure 1.

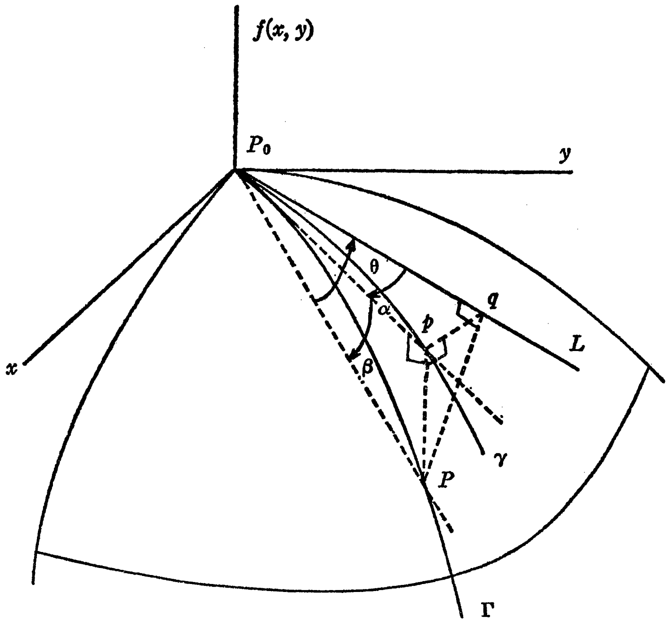

FIG. 1.

If q is the projection of P on L, then we have:

$$(|P_0P| \sin \theta)^2 = (|P_0P| \sin \beta)^2 + (|P_0p|^2 + |P_0q|^2 - 2|P_0p|\,|P_0q| \cos \alpha)$$

which reduces to:

$$\cos \theta = \cos \alpha \cos \beta. \tag{1}$$

Since θ approaches 0 as P approaches P_0, so does β. If we define $\beta(x, y)$ to be the angle between P_0P and the plane $z = 0$ for each $P \neq P_0$ and $\beta(0, 0) = 0$, we see that $\beta(x, y)$ is continuous at $(0, 0)$ for every curve under discussion. Since convex curves are included, the following theorem of Rosenthal [6, p. 31] applies and $\beta(x, y)$ is continuous at $(0, 0)$. Hence, $z = 0$ is a tangent plane under Definition II.

LEMMA I. *If the single valued function* β *is continuous at* p_0 *along every convex curve through* p_0 *which is (at least) once differentiable, then* β *is also continuous at* p_0 *as a function of* (x, y).

(II→I). Suppose there exists a tangent plane under Definition II which we assume to be $z=0$. Then the angle β function above has limit zero as P approaches P_0. Therefore β approaches zero on every differentiable curve γ. Let L be the tangent line to γ at (0, 0). L is in $z=0$. Angle α approaches zero as p approaches (0, 0) and therefore as P approaches P_0. By equation (1), angle ϕ also approaches zero as P approaches P_0 and hence L is also the tangent line to curve Γ. Thus these tangent lines all exist and lie in the plane $z=0$ which is then a tangent plane under Definition I.

Remarks. Lemma I suggests that an "apparent weakening" of Definition I restricting the curves γ to be convex and once differentiable is possible. An alternative approach to the demonstration of the equivalence of these definitions is to show each to be equivalent to the differentiability of f at P_0. The proof of this for Definition II may be found in [**5**, p. 217]. The proof for Definition I can be shown using the following theorem of Krishnaiah and Rao [**7**].

LEMMA 2. *Suppose* $f_x(0, 0)=f_y(0, 0)=0$. *A necessary and sufficient condition for* f *to be differentiable at* (0, 0) *is that, for every pair* $(g(t), h(t))$ *of functions, differentiable at* 0, *such that* $g(0)=h(0)=0$, *the derivative of* $f(g(t))$ *exists and vanishes at* 0.

It should not be surprising that the proof given in [**7**] for Lemma 2 does require Lemma 1.

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SCHWARZ DIFFERENTIABILITY AND DIFFERENTIABILITY

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Let $f(x)$ be a real function of the real variable x defined in a neighborhood of a point a. If the limit

$$\lim_{h\to 0}\frac{f(a+h)-f(a-h)}{2h}$$

LEMMA I. *If the single valued function β is continuous at p_0 along every convex curve through p_0 which is (at least) once differentiable, then β is also continuous at p_0 as a function of (x, y).*

(II→I). Suppose there exists a tangent plane under Definition II which we assume to be $z=0$. Then the angle β function above has limit zero as P approaches P_0. Therefore β approaches zero on every differentiable curve γ. Let L be the tangent line to γ at (0, 0). L is in $z=0$. Angle α approaches zero as p approaches (0, 0) and therefore as P approaches P_0. By equation (1), angle ϕ also approaches zero as P approaches P_0 and hence L is also the tangent line to curve Γ. Thus these tangent lines all exist and lie in the plane $z=0$ which is then a tangent plane under Definition I.

Remarks. Lemma I suggests that an "apparent weakening" of Definition I restricting the curves γ to be convex and once differentiable is possible. An alternative approach to the demonstration of the equivalence of these definitions is to show each to be equivalent to the differentiability of f at P_0. The proof of this for Definition II may be found in [**5**, p. 217]. The proof for Definition I can be shown using the following theorem of Krishnaiah and Rao [**7**].

LEMMA 2. *Suppose $f_x(0, 0)=f_y(0, 0)=0$. A necessary and sufficient condition for f to be differentiable at (0, 0) is that, for every pair $(g(t), h(t))$ of functions, differentiable at 0, such that $g(0)=h(0)=0$, the derivative of $f(g(t))$ exists and vanishes at 0.*

It should not be surprising that the proof given in [**7**] for Lemma 2 does require Lemma 1.

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SCHWARZ DIFFERENTIABILITY AND DIFFERENTIABILITY

SIMEON REICH, Student, Israel Institute of Technology

Let $f(x)$ be a real function of the real variable x defined in a neighborhood of a point a. If the limit

$$\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h}$$

exists and is finite, we say that $f(x)$ is Schwarz differentiable at a. The Schwarz derivative (also called the symmetric derivative) will be denoted by $f^s(x)$.

Let I be a neighborhood of a of which $[c, d]$, $c<a<d$, is a subset. Let $f(x)$ be Schwarz differentiable at each point x of I and put

$$\psi(x, h) = [f(x+h) - f(x-h)]/(2h) - f^s(x).$$

Then $f(x)$ is said to be uniformly Schwarz differentiable in $[c, d]$, or more informally, uniformly Schwarz differentiable in a neighborhood of a, if corresponding to an arbitrary $\epsilon>0$, there exists a $\delta>0$ (independent of x) such that $|\psi(x, h)| < \epsilon$, if $0<|h|<\delta$ for $x\in[c, d]$ and $x\pm h\in I$. The concept of uniform Schwarz differentiability is due to Mukhopadhyay.

On the one hand, [1] shows that the continuity of $f^s(x)$ in a neighborhood of a and the continuity of $f(x)$ at a do not imply the existence of the ordinary derivative at a. On the other hand, Theorem 3 of [2] demonstrates that the continuity of $f^s(x)$ at a and the continuity of $f(x)$ in a neighborhood of a do imply the existence of $f'(x)$ at a. Thus the following two simple results may be of interest:

THEOREM 1. *If $f^s(x)$ is continuous at a and if $f(x)$ is uniformly Schwarz differentiable in a neighborhood of a, then $f'(a)$ exists.*

Proof. Let $\epsilon>0$ be given. By the uniform Schwarz differentiability of $f(x)$ we have

$$|[f(a+t+h) - f(a+t-h)]/(2h) - f^s(a+t)| < \epsilon/2$$

for all $|t|<\delta_1>0$ and for all $0<|h|<\delta_2>0$. By the continuity of $f^s(x)$ at a we have

$$|f^s(a+t) - f^s(a)| < \epsilon/2$$

for all $|t|<\delta_3>0$. It follows that

$$\begin{aligned} &|[f(a+t+h) - f(a+t-h)]/(2h) - f^s(a)| \\ &\quad \leqq |[f(a+t+h) - f(a+t-h)]/(2h) - f^s(a+t)| \\ &\quad\quad + |f^s(a+t) - f^s(a)| < \epsilon/2 + \epsilon/2 = \epsilon \quad \text{for } 0 < |t|, |h| < \delta/2 \end{aligned}$$

where $\delta = 2\min(\delta_1, \delta_2, \delta_3)$. Putting $t=h$ we obtain

$$|[f(a+2h) - f(a)]/(2h) - f^s(a)| < \epsilon, \quad 0 < |h| < \delta/2.$$

Finally, denoting $2h$ by H we get

$$|[f(a+H) - f(a)]/H - f^s(a)| < \epsilon, \ 0 < |H| < \delta.$$

That is, $f'(a)$ exists (and it equals $f^s(a)$).

In fact, we could have established this theorem by appealing to a result of Mukhopadhyay. This will be done in the proof of,

THEOREM 2. *If $f(x)$ is continuous at a and uniformly Schwarz differentiable in a neighborhood of a, then $f'(x)$ exists at a.*

Proof. Since $f(x)$ is continuous at a, it is bounded (by M) in some neighborhood I which may be assumed to be contained in the neighborhood of the statement of the theorem, and which contains an interval $[c, d]$, $c<a<d$, in which $f(x)$ is uniformly Schwarz differentiable. There exists an $h_0>0$ such that $|\psi(x, h_0)|<\epsilon_0$ for all $x\in[c, d]$, and for some $\epsilon_0>0$. Hence

$$|f^s(x)| < \epsilon_0 + |[f(x+h_0) - f(x-h_0)]/(2h_0)| \leqq \epsilon_0 + M/h_0$$

for all $x\in[c, d]$. By Theorem 1 of [3] $f(x)$ is continuous in $[c, d]$. By Result A of [3] $f^s(x)$ is continuous in some subinterval $[c_1, d_1]$, $c_1<a<d_1$, of $[c, d]$. By Theorem 2 of [2], $f'(a)$ exists, as required.

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ELLIPSE OF LEAST ECCENTRICITY

ROGER D. H. JONES, The University of Georgia

In this MAGAZINE, vol. 39, no. 4 (Sept. 1966), pp. 203–205, Vinh considered a geometrical problem that has an application to astronautics. The problem was (see Figure 1) to minimize $OF/(OP+PF)$, where O, P, and line PF are fixed. Vinh gave two solutions, a trigonometric one and a geometric one. Here is a third (very simple geometric) solution.

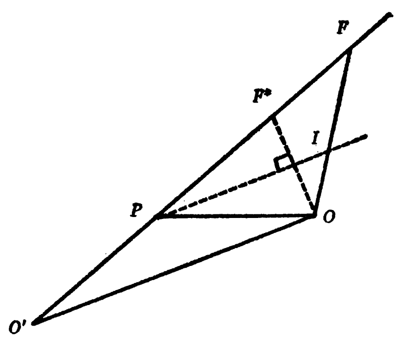

FIG. 1.

Produce FP to O' such that $PO'=PO$. Then the problem of minimizing $OF/(OP+PF)$ is the same as that of minimizing $OF/O'F$. But

$$OF/O'F = \sin(FO'O)/\sin(FOO').$$

Proof. Since $f(x)$ is continuous at a, it is bounded (by M) in some neighborhood I which may be assumed to be contained in the neighborhood of the statement of the theorem, and which contains an interval $[c, d]$, $c<a<d$, in which $f(x)$ is uniformly Schwarz differentiable. There exists an $h_0>0$ such that $|\psi(x, h_0)| <\epsilon_0$ for all $x\in[c, d]$, and for some $\epsilon_0>0$. Hence

$$|f^s(x)| < \epsilon_0 + |[f(x+h_0) - f(x-h_0)]/(2h_0)| \leqq \epsilon_0 + M/h_0$$

for all $x\in[c, d]$. By Theorem 1 of [3] $f(x)$ is continuous in $[c, d]$. By Result A of [3] $f^s(x)$ is continuous in some subinterval $[c_1, d_1]$, $c_1<a<d_1$, of $[c, d]$. By Theorem 2 of [2], $f'(a)$ exists, as required.

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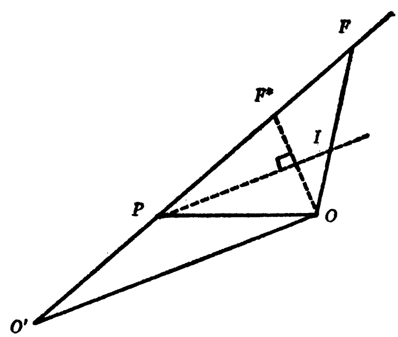

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Now $\measuredangle FO'O$ is fixed and hence this ratio is least when $\measuredangle FOO'=90°$. PI bisects $\measuredangle FPO$ and triangle $PO'O$ is isosceles; hence PI is parallel to $O'O$. But FO is proved above to be perpendicular to $O'O$ when the ratio $OF/O'F$ is a minimum. Therefore FO is perpendicular to PI when this ratio is a minimum, which is the conclusion of Vinh on p. 205.

Editorial Note: Professor P. N. Bajaj, Wichita State University, has submitted two easy solutions of this problem, one using elementary algebra and the other using simple calculus.

ON THE L'HÔPITAL RULE FOR INDETERMINATE FORMS ∞/∞

MEN-CHANG HU, National Taiwan University and JU-KWEI WANG, University of Massachusetts

The l'Hôpital rule which we are concerned with is also known as the Northeast Theorem, and it is stated as follows:

Let f and g be differentiable functions near $x=a$. If

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$$

and

$$\lim_{x\to a} f'(x)/g'(x) = l,$$

then $\lim_{x\to a} f(x)/g(x)$ *exists and is equal to l.*

We think the following proof is simple enough to be presented in most elementary calculus courses where the logarithm is introduced early.

Proof. We may assume without loss of generality that $l>0$, for otherwise we may use

$$f_1(x) = f(x) + cg(x)$$

instead of $f(x)$ for a large enough c.

Let $\epsilon>0$ be given. There exists $\eta>0$ such that if $0<|x-a|<\eta$, then

$$|\ln[f'(x)/g'(x)] - \ln l| < \tfrac{1}{2}\epsilon.$$

Select x_1 such that $0<|x_1-a|<\eta$ and fix it. Then there exists $\eta'<\eta$ such that if $0<|x-a|<\eta'$, then $f(x)>f(x_1)$ and $g(x)>g(x_1)$. Then

$$\begin{aligned} |\ln[f(x)/g(x)] - \ln l| &= \left|\ln\frac{f(x)-f(x_1)}{g(x)-g(x_1)} - \ln l - \ln\frac{1-f(x_1)/f(x)}{1-g(x_1)/g(x)}\right| \\ &\leqq \left|\ln\frac{f'(\xi)}{g'(\xi)} - \ln l\right| + \left|\ln\frac{1-f(x_1)/f(x)}{1-g(x_1)/g(x)}\right|, \end{aligned}$$

Now $\measuredangle FO'O$ is fixed and hence this ratio is least when $\measuredangle FOO' = 90°$. PI bisects $\measuredangle FPO$ and triangle $PO'O$ is isosceles; hence PI is parallel to $O'O$. But FO is proved above to be perpendicular to $O'O$ when the ratio $OF/O'F$ is a minimum. Therefore FO is perpendicular to PI when this ratio is a minimum, which is the conclusion of Vinh on p. 205.

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where ξ lies between x and x_1. The first term is less than $\frac{1}{2}\epsilon$ by our choice of η. Since the logarithm is a continuous function and since $\ln 1=0$, we may choose $\delta<\eta'$ such that if $0<|x-a|<\delta$, then

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Then

$$|\ln[f(x)/g(x)]-\ln l|<\epsilon.$$

This means that

$$\lim_{x\to a}\ln[f(x)/g(x)]=\ln l,$$

or

$$\lim_{x\to a}f(x)/g(x)=l.$$

This completes the proof.

ON RECONSTRUCTION OF MATRICES

BENNET MANVEL, Colorado State University and
PAUL K. STOCKMEYER, University of Michigan and The College of William and Mary

Considerable attention has been devoted to the problem of reconstructing a graph from its maximal subgraphs. This problem seems to be very difficult in the general case, although certain classes of graphs have been shown to be reconstructable (see the references). We deal here with a similar problem for matrices, which has some graph theoretic consequences.

If $A=(a_{i,j})$ is any $n\times n$ matrix, we call the submatrix obtained by deletion of its kth row and kth column the *kth principal minor*, and denote it by A_k. We are interested in showing that the principal minors determine the matrix. That is of course trivially obvious if it is known which minor is first, which is second, and so forth. Since such knowledge is implied by the notation A_k, we introduce a new collection of matrices B_k which are just the matrices A_k in an arbitrary order. That is, there exists a permutation σ such that $B_{\sigma i}=A_i$, $1\leqq i\leqq n$. The i,j entry of B_k will be denoted by $b_{i,j;k}$.

THEOREM. *Any $n\times n$ matrix A with $n\geqq 5$ can be reconstructed from its list of principal minors. Moreover, there exist matrices which require at least $-[-n/2]+2$ such minors for reconstruction.*

Proof. To reconstruct A we first find the diagonal entries by noting that the list of entries $b_{i,i;k}$ contains $n-i$ copies of $a_{i,i}$ and i copies of $a_{i+1,i+1}$. In the absence of any standard notation, we write this in the following way:

where ξ lies between x and x_1. The first term is less than $\frac{1}{2}\epsilon$ by our choice of η. Since the logarithm is a continuous function and since $\ln 1=0$, we may choose $\delta<\eta'$ such that if $0<|x-a|<\delta$, then

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Then

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$$\{b_{i,i;k} \mid 1 \leqq k \leqq n\} = (n-i)\{a_{i,i}\} + i\{a_{i+1,i+1}\}. \tag{1}$$

Thus for $n \geqq 3$, we can find $a_{1,1}$ and, having that, we can proceed inductively to find the other diagonal entries.

The job of finding $a_{i,j}$, $i<j$, is slightly more difficult. The i, j entries of the B_k are related to the entries of A as follows:

$$\{b_{i,j;k} \mid 1 \leqq k \leqq n\} = (n-j)\{a_{i,j}\} + (j-i)\{a_{i,j+1}\} + i\{a_{i+1,j+1}\}. \tag{2}$$

If n is at least 5, then the number which appears most often in the list of $b_{1,2;k}$ must be $a_{1,2}$, since $n-j \geqq 3$ and $j-i=i=1$.

If $n \geqq 7$ we can determine $a_{1,3}$, $a_{1,4}$, and $a_{2,4}$ from equation (2) with $i=1, j=3$ in a similar way. This is useful since once we know $a_{1,3}$ we can repeatedly apply equation (2) specialized to $i=1$ to find the entire first row of A. Once we know that, it is obvious that the other $a_{i,j}$ can be found using equation (2). The $a_{i,j}$ for $i>j$ can be obtained in a similar way.

For $n=5$ and $n=6$ special arguments are needed to determine the entry $a_{1,3}$. Setting $n=6$ we have the following cases of equation (2):

$$\{b_{1,2;k} \mid 1 \leqq k \leqq 6\} = 4\{a_{1,2}\} + \{a_{1,3}\} + \{a_{2,3}\} \tag{3}$$

$$\{b_{1,3;k} \mid 1 \leqq k \leqq 6\} = 3\{a_{1,3}\} + 2\{a_{1,4}\} + \{a_{2,4}\} \tag{4}$$

$$\{b_{2,3;k} \mid 1 \leqq k \leqq 6\} = 3\{a_{2,3}\} + \{a_{2,4}\} + 2\{a_{3,4}\}. \tag{5}$$

If $a_{1,3}=a_{2,3}$, we can find $a_{1,3}$ from (3). If not, we can solve equation (4) for $a_{1,3}$ unless $a_{1,4}=a_{2,4} \neq a_{1,3}$. In that case we know only the sets $\{a_{1,3}, a_{2,3}\}$ and $\{a_{1,3}, a_{2,4}\}$. If $a_{2,3} \neq a_{2,4}$, then $a_{1,3} = \{a_{1,3}, a_{2,3}\} \cap \{a_{2,4}, a_{1,3}\}$; if $a_{2,3}=a_{2,4}$, we can obtain their common value from (5) and then derive $a_{1,3}$ from equation (3). The argument for $n=5$ is similar but more involved and is omitted.

The matrices $M=(m_{i,j})$ and $N=(n_{i,j})$ defined below have $-[-n/2]+1$ submatrices in common and hence prove the second part of the theorem.

$$m_{i,j} = \begin{cases} 1 & \text{if } i=1 \text{ and } j = 1+[n/2] \\ 1 & \text{if } j=1 \text{ and } i = 1+[n/2] \\ 0 & \text{otherwise} \end{cases}$$

$$n_{i,j} = \begin{cases} 1 & \text{if } i=1 \text{ and } j = 2+[n/2] \\ 1 & \text{if } j=1 \text{ and } i = 2+[n/2] \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see that 2×2 and 3×3 matrices cannot always be reconstructed in this way. The following two 4×4 matrices have the same list of principal submatrices, and hence cannot be reconstructed:

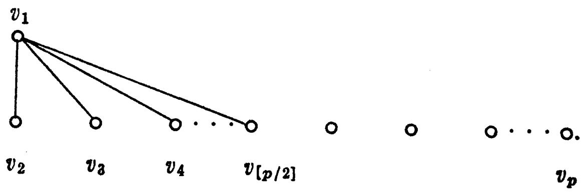
$$\begin{bmatrix} b & d & c & d \\ e & b & c & c \\ f & f & b & d \\ e & f & e & b \end{bmatrix} \text{ and } \begin{bmatrix} b & c & d & c \\ f & b & d & d \\ e & e & b & c \\ f & e & f & b \end{bmatrix}.$$

Because of the correspondence between graphs and their adjacency matrices (for graphical definitions see [5]) our theorem has the following graph-theoretic corollary, which could also be stated for pseudographs or directed graphs.

COROLLARY. *Any graph G with $p \geqq 5$ points which are labeled $v_1, \cdots, v_p$ can be reconstructed from its maximal subgraphs $G_i = G - v_i$, if their points are labeled $u_1, \cdots, u_{p-1}$ in the same order. Furthermore, there are graphs which require at least $[p/2]+2$ such subgraphs for reconstruction.*

The difference between the bound of the theorem and that of its corollary stems from the fact that the matrices M and N given in the proof of the theorem are similar and hence correspond to the same graph. The following pair of graphs have $[p/2]+1$ subgraphs G_i in common, with their points ordered as required.

G:



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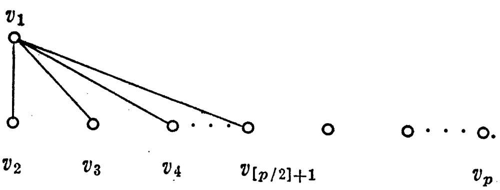

A more precise determination of the number of submatrices actually required to reconstruct an arbitrary $n \times n$ matrix appears to be difficult.

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ON HUYGENS' APPROXIMATION TO π

T. S. NANJUNDIAH, University of Mysore, India

Consider the arithmetic and harmonic means

$$a_n = \frac{P_n + 2p_n}{3}, \qquad h_n = \frac{3}{1/P_n + 2/p_n}$$

of the triple $\{P_n, p_n, p_n\}$, where $P_n = n \tan \pi/n$, $p_n = n \sin \pi/n$ are the perimeters of the regular n-gons circumscribed to and inscribed in the circle whose circumference is π. A simple and fairly good approximation to π due to Huygens [1] is $\pi \approx a_n$ in which the error satisfies

$$(1_n) \qquad 15n^{-4} < a_n - \pi < 27n^{-4}.$$

A closer approximation, however, is $\pi \approx h_n$ with the error satisfying

$$(2_n) \qquad \frac{5}{3}n^{-4} < \pi - h_n < 2n^{-4}.$$

Actually, π *lies in the first* 10*th part of the interval* (h_n, a_n). But even more is true:

$$(3_n) \qquad h_n < \pi < h_n^* = \frac{10}{9/h_n + 1/a_n}.$$

This points to the approximation $\pi \approx h_n^*$, which is much better than the previous one, as the error in it satisfies

$$(4_n) \qquad \frac{7}{2}n^{-6} < h_n^* - \pi < 6n^{-6}.$$

The facts asserted above are just special cases of interesting estimates for $\sin x/x$ by *rational* functions of $\cos x$, it being assumed throughout that $0 < x < \pi/2$. Thus, introducing the arithmetic and harmonic means

$$a(x) = \frac{\cos x + 2}{3}, \qquad h(x) = \frac{3 \cos x}{1 + 2 \cos x}$$

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ON HUYGENS' APPROXIMATION TO π

T. S. NANJUNDIAH, University of Mysore, India

Consider the arithmetic and harmonic means

$$a_n = \frac{P_n + 2p_n}{3}, \qquad h_n = \frac{3}{1/P_n + 2/p_n}$$

of the triple $\{P_n, p_n, p_n\}$, where $P_n = n \tan \pi/n$, $p_n = n \sin \pi/n$ are the perimeters of the regular n-gons circumscribed to and inscribed in the circle whose circumference is π. A simple and fairly good approximation to π due to Huygens [1] is $\pi \approx a_n$ in which the error satisfies

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The facts asserted above are just special cases of interesting estimates for $\sin x/x$ by *rational* functions of $\cos x$, it being assumed throughout that $0 < x < \pi/2$. Thus, introducing the arithmetic and harmonic means

$$a(x) = \frac{\cos x + 2}{3}, \qquad h(x) = \frac{3 \cos x}{1 + 2 \cos x}$$

of the triple $\{\cos x, 1, 1\}$, we have

(1) $$x^{-5}\left(\frac{\sin x}{h(x)} - x\right) \downarrow \frac{1}{20} \qquad \text{as } x \downarrow 0,$$

(2) $$x^{-5}\left(x - \frac{\sin x}{a(x)}\right) \downarrow \frac{1}{180} \qquad \text{as } x \downarrow 0,$$

(3) $$a^*(x) = \frac{9a(x) + h(x)}{10} < \frac{\sin x}{x} < a(x),$$

(4) $$x^{-7}\left(\frac{\sin x}{a^*(x)} - x\right) \downarrow \frac{1}{840} \qquad \text{as } x \downarrow 0.$$

The special cases of these in which $x = \pi/n$ $(n \geqq 3)$ are easily seen to imply (1_n), (2_n), (3_n) and (4_n) respectively. For the proofs of (1), (2) and (4), barring a routine calculation of the limits, we merely note from the results of straightforward differentiation that the derivatives of the functions on the left have, in order, the signs of

$$x - \frac{5sc(1 + 2c)}{1 + 12c^2 + 2c^3}, \qquad \frac{15s(2 + c)}{19 + 22c + 4c^2} - x$$

and

$$x - \frac{35s(1 + 2c)(1 + 3c + c^2)}{23 + 123c + 228c^2 + 133c^3 + 18c^4},$$

where $s = \sin x$ and $c = \cos x$. These functions all vanish for $x = 0$, so that they are positive with their corresponding derivatives

$$\frac{2(1 - c)^3(3 + 19c + 25c^2 - 2c^3)}{(1 + 12c^2 + 2c^3)^2}, \qquad \frac{2(1 - c)^3(7 + 8c)}{(19 + 22c + 4c^2)^2}$$

and

$$\frac{(1 - c)^4(249 + 1159c + 1809c^2 + 1184c^3 + 324c^4)}{(23 + 123c + 228c^2 + 133c^3 + 18c^4)^2}.$$

Of course, (3) is covered by (2) and (4). But it follows more readily by observing that the functions

$$x - \frac{\sin x}{a(x)}, \qquad \frac{\sin x}{a^*(x)} - x$$

vanish for $x = 0$ and so are positive with their respective derivatives

$$\left(\frac{1 - c}{2 + c}\right)^2, \qquad \frac{(1 - c)^3(2 + 3c)}{3(1 + 3c + c^2)^2}.$$

We restate the result in words: *for* $0 < x < \pi/2$, $\sin x/x$ *lies in the last* 10*th part of the interval* $(h(x), a(x))$.

Reference

1. N. M. Günter and R. O. Kusmin, Aufgabensammlung zur höheren Mathematik, Bd.I, Berlin, 1960, p. 123.

BOOK REVIEWS

EDITED BY D. ELIZABETH KENNEDY, University of Victoria

Materials intended for review should be sent to: Professor D. Elizabeth Kennedy, Department of Mathematics, University of Victoria, Victoria, British Columbia, Canada.

Reviews of texts at the freshman-sophomore level based upon classroom experience will be welcomed by the Book Review Editor.

A boldface capital **C** *in the margin indicates a classroom review.*

C *A Survey of Finite Mathematics.* By Marvin Marcus. Houghton-Mifflin, Boston, 1969. ix+486 pp.

Mathematics for the Social and Management Sciences, Finite Mathematics. By Guillermo Owen. W. B. Saunders Company, Philadelphia, 1970. xi+424 pp.

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C *Finite Mathematics: A Liberal Arts Approach.* By Irving Allen Dodes. McGraw-Hill, New York, 1970. viii+403 pp.

Until recently the choices for texts in the finite mathematics course were few. Some of the most widely used books were those of Kemeny, Snell, and Thompson (hereafter referred to as KST), Katsoff and Simone, and the *Schaum's Outline* by S. Lipschutz. The situation has changed with the appearance of the books under review and others, so that there is now something for everyone. The four books are quite distinct, both in the level of difficulty (they are listed in decreasing order) and in the philosophy of approach.

A Survey by Marcus has three lengthy chapters, **Fundamentals, Linear Algebra,** and **Convexity.** The first chapter treats logic, set theory, functions, combinatorics, partitions, probability, and finite stochastic processes. It is to be covered in about 30 lectures. The material in Chapter II gives a very good introduction to the usual topics of linear algebra and provides as well many interesting applications to incidence matrices, relations, and combinatorial matrix theory. Many proofs here would be out of reach of students the reviewer has known in this course. Chapter III opens with linear programming and geometry in Euclidean spaces (inner products, CBS inequality, bounded sets, distance from a point to a hyperplane). Terse but complete treatments of game theory and linear programming are given as applications of the theory of linear functions on convex sets. Professor Marcus gives his usual clear presentation with impeccable logic throughout. Theorems, definitions, and examples are clearly labeled, and proofs are included for all theorems. There are more than

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150 interesting applications, but more routine examples may be necessary for the average student. The applications seem more "relevant" than those which usually appear in such texts. True-false quizzes are found at the end of each of the 19 sections of the book, and there are more than 1200 exercises with answers and discussions of solutions provided for just fewer than half. The level and style of this book make it appealing for use by math majors. Quantifiers are used in the coverage of logic, but they only appear sparingly in the subsequent text. The treatment of combinatorics, a speciality of the author, is much deeper than usual, and mathematical induction is used freely. Some of the uses of *sigma* and *pi* notation, as well as the proliferation of sub- and superscripts may bewilder some students (e.g., see Chapter I, Theorems 4.1, 5.2, 6.2). On pages 63–64 is found the definition of a category of sets (though it is nowhere used), and the only reference of any kind in the book is to the Proceedings of the National Academy of Science, where one may learn how: **the usual axioms about sets can be replaced by axioms concerning categories.** A finite probability space is defined in terms of a *sigma* field (using German script notation) and a probability measure, while the "tree" approach to finite stochastic processes is through product spaces of n probability spaces. It should be clear that this excellent but ambitious text should be chosen only after careful consideration of the abilities of the students who will use it. The book has been used at Valparaiso University, and Ruth K. Deters reports that, although the development in this text is somewhat more rigorous than that of its competitors (including KST), it was found satisfactory for a semester course for selected freshmen through seniors (nonmajors). Chapter I, four sections of Chapter II, and the sections covering linear programming and Markov chains from Chapter III were used. Most of the students had previously taken a one semester calculus course.

The second book under review also strongly reflects the research interests of its author. There are eight chapters, and it would seem to be a two-semester text. Chapter 1 covers analytic geometry, linear equations, and inequalities, first in the plane and then in higher dimensions. Slope is discussed in terms of the tangent function. Chapter 2 is a standard approach to vectors, matrices, and the solution of systems of equations. Linear programming is then given a full treatment, including the simplex method (complete proof and discussion of degeneracy), duality, transportation problems, and assignment problems. A short chapter on combinatorics (chiefly combinations) also contains some material on truth tables. The chapter on discrete probability is similar to that of KST in its approach, but some statistics theory is also presented. The chapter on Markov chains is very compact, but it, together with the accompanying problems, covers a great deal of ground. The last three chapters are distinctive. Game theory is the author's research interest, and this is reflected in the very complete coverage of that subject given here. Detailed discussions are given of the von Neumann model of an expanding economy, nonzero-sum games, n-person games, the Shaply value, and voting structures. There follows a chapter on dynamic programming (transportation and inventory problems), and the book closes with a chapter on graphs and networks, the emphasis being on shortest paths and maximal flows. This material has not appeared in such a text before. Many teachers who are familiar only with the KST brand of finite mathematics will find much of the material in this book new and interesting since the slant is toward operations research. The exercises are mostly routine and computational, and answers are not given. An instructor's guide is available. There are useful appendices on solution of equations, induction, exponents and logarithms, and *sigma* notation.

Goodman and Ratti state in their preface that: **The course in finite mathematics is now well established. The guiding principle is to present to the reader a slice of mathematics that is interesting, meaningful, and useful and that at the same time does not involve the calculus. With these limitations the content of a text for the course is almost uniquely determined.** Thus, it is not surprising that the topics covered in their book are substantially those of KST in the same order as KST. The book is written at a level slightly more accessible to average students, and more background material is included. There is a preliminary chapter covering mathematical symbols, subscripts, prime numbers, absolute value, and some other minor items, and there are appendices on functions, inequalities, and mathematical induction. The book is well written and the production is very attractive. Answers to about two thirds of the problems are given.

In contrast to the preceding quote, Dodes states in his preface that: **'Finite mathematics' is not very well defined,** and he goes on to lament the fact that: **most *courses* in finite mathematics do not get beyond the first half of books on the subject.** Thus, he has: **cut to a minimum all those topics that are mainly of interest to a mathematics major . . . . My purpose was to get to topics more relevant to a liberal arts or business student . . . . No assumptions have been made about the previous training of the students.** The level and character of this book are consequently quite different from the others. The first chapter, **An Introduction to Operations Research,** contains a short discussion of sets (not really used in the rest of the text), applications to incomplete polls, graphs of lines and polygons in the plane, and linear programming in the plane. No discussion of the higher dimensional cases is given, and the problems are all designed so as not to require a general discussion of solution of linear systems. **The Mathematics of Gambling** is the title of the second chapter. This is a sugar-coated approach to probability theory without much theory. The problems are all involved with dice, coin tossing, and card shuffling. Conditional probabilities are not introduced, and combinatorics enter the scene only though *ad hoc* discussions. Trees (here called Kemeny-trees) are used throughout. Elementary material on vectors and matrices is presented in the next chapter, and applications are made to transitions and Markov chains, but examples and problems only go as far as two or three transitions. All column vectors are treated as transposes of row vectors. The next two chapters, comprising 75 pages of the text, cover game theory. There is more material here than could sustain the interest of our students. There is little justification given for just why this material should be mastered, and after the initial flush of victory over saddle-point games and two-by-two games with mixed strategy, the students lost interest in the tedious treatment of two-by-m and three-by-three games. The next three chapters, mostly skipped in our semester course at Loyola, contain material on the slide rule, logarithms, growth and decay curves, semilog paper, and some elementary cookbook-style math of finance. The material on growth and decay could be very useful to a student in the soft sciences, but the slant here is toward the physical sciences. Radioactive decay and electric transients seem to have little appeal to the liberal arts student. The next chapter is **The Mathematics of Management.** Students enjoyed the drunkard's walk (although I doubt that many appreciated the discussion of Brownian motion—why not use this material to simulate an epidemic or the spread of a rumor?), as well as the simulation of a supermarket checkout counter, and PERT charts. We did not use the portion of this chapter of "forecasting," which includes trends, seasonal variation, and smoothing of data. The book concludes with a chapter on computing and FORTRAN. After some binary and hexadecimal arithmetic and a discussion of computer hardware there is a 22 page introduction to FORTRAN without FORMAT or DO statements—not enough to write a program that will run. Since we had a compiler with free-format-input-output available, we were able to supplement this chapter so that we could require the students to write and run some simple programs. While some exposure to the computer should be a part of a liberal arts finite mathematics course, there are still many problems in getting the appropriate material into a text for this course.

Professor Dodes has taken obvious care to write a book which is understandable to all. He uses a chatty, glib style which sometimes leaves a bad taste. Some of the better students felt that the author was talking down to them, and a very unfortunate example of his style is this quote: **A game with a value of zero is called a fair game or (this is clever!) a zero-sum game.** This may be clever, but it is also incorrect. There are many examples and a sufficient number of problems in this text, but some border on the trivial. A very useful instuctor's manual is available. Despite the misgivings of the instructors who used this text at Loyola, it was generally well received by the students.

In summary, each of these books will be useful if it is chosen for the right clientele. The first (or third) books listed could well be used in lieu of KST if the teacher wishes to raise (or lower) his sights a bit while keeping the character of the course the same. If one wishes to infuse some different material, or if the students are business majors of high ability, then Owen's book is appropriate. The Dodes book is useful in a Math for Poets course or in a terminal course for students with weak mathematical backgrounds.

G. C. DORNER, Loyola University of Chicago

CUPM REPORT: A COURSE IN BASIC MATHEMATICS FOR COLLEGES

In January, 1970, the Committee on the Undergraduate Program in Mathematics (CUPM) initiated a study concerning curricular problems for those students who are deficient in basic mathematics. There is a sizeable number of chapters, mostly skipped in our semester course at Loyola, contain material on the slide rule, logarithms, growth and decay curves, semilog paper, and some elementary cookbook-style math of finance. The material on growth and decay could be very useful to a student in the soft sciences, but the slant here is toward the physical sciences. Radioactive decay and electric transients seem to have little appeal to the liberal arts student. The next chapter is **The Mathematics of Management.** Students enjoyed the drunkard's walk (although I doubt that many appreciated the discussion of Brownian motion—why not use this material to simulate an epidemic or the spread of a rumor?), as well as the simulation of a supermarket checkout counter, and PERT charts. We did not use the portion of this chapter of "forecasting," which includes trends, seasonal variation, and smoothing of data. The book concludes with a chapter on computing and FORTRAN. After some binary and hexadecimal arithmetic and a discussion of computer hardware there is a 22 page introduction to FORTRAN without FORMAT or DO statements—not enough to write a program that will run. Since we had a compiler with free-format-input-output available, we were able to supplement this chapter so that we could require the students to write and run some simple programs. While some exposure to the computer should be a part of a liberal arts finite mathematics course, there are still many problems in getting the appropriate material into a text for this course.

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It is proposed that some of the currently existing basic mathematics courses be replaced by this flexible one-year course, together with an accompanying mathematics laboratory. The laboratory would serve to remedy the students' arithmetic deficiencies, offer added opportunity for drill in algebraic manipulations and allow for instruction in several vocational-oriented topics. The main aim of this course will be to provide the students with enough mathematical literacy for adequate participation in the daily life of our present society.

Many of the students in standard basic mathematics courses have seen the same material in elementary and secondary schools, and it is often the case that this second exposure is no more successful than the first. Thus, a new and more appropriate approach is needed to meet the needs of college students.

In Mathematics E it is recommended that flow-charting and algorithmic and computer-related ideas be introduced early and used throughout. This should give the student a technique in the analysis of problems and encourage him to be precise in dealing with both arithmetic and nonarithmetic operations. Topics of everyday concern, such as how bills are prepared by a computer, calculation of interest in installment buying, quick estimation, analyses of statistics appearing in the press, and various job-related algebraic and geometric problems, are mainstays of the course.

In order to make the recommendations as clear as possible, a topical outline, with an extensive commentary, is given. However, the outline should be viewed more as a flexible model rather than a rigid description; the spirit of the course is more important than content. The model outline contains flow charts and elementary operations, rational numbers, geometry I, linear polynomials and equations, the computer, nonlinear relationships, geometry II, statistics, and probability.

A Course in Basic Mathematics for Colleges is available without charge from CUPM, P.O. Box 1024, Berkeley, California 94701.

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles Valley College

ASSOCIATE EDITOR, J. SUTHERLAND FRAME, Michigan State University

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Problems may be submitted from any branch of mathematics and ranging in subject content from that accessible to the talented high school student to problems challenging to the professional mathematician. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

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The asterisk (*) will be placed by the problem number to indicate that the proposer did not supply a solution. Readers' solutions are solicited for all problems proposed. Proposers' solutions may not be "best possible" and solutions by others will be given preference.

Solutions should be legible and submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction.

Send all communications for this department to Robert E. Horton, Los Angeles Valley College, 5800 Fulton Avenue, Van Nuys, California 91401.

To be considered for publication, solutions should be mailed before January 15, 1972.

PROPOSALS

803. *Proposed by Kenneth Rosen, University of Michigan.*

Let x and y be positive real numbers with $x+y=1$. Prove that $x^x+y^y \geqq \sqrt{2}$ and discuss conditions for equality.

804.* *Proposed by Zalman Usiskin, University of Chicago.*

Forty golfers play each week ten foursomes. Thus in thirteen weeks it is possible for a particular golfer to play every other golfer. Is it possible for every golfer to play in a foursome with every other golfer in this minimal length of time?

805. *Proposed by Charles W. Trigg, San Diego, California*

Find the unique triangular number Δ_n which is a permutation of the ten digits and for which n has the form *abbbb*.

806. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let H be the orthocenter of an isosceles triangle ABC, and let AH, BH, and CH intersect the opposite sides in D, E, and F, respectively. Prove that the incenters of the right triangles HBD, HDC, HCE, HEA, HAF, and HFB lie on a conic.

807. *Proposed by Norman Schaumberger, Bronx Community College.*

Let (x_i), $i=1, 2, 3 \cdots$ be an arbitrary sequence of positive real numbers, and set

$$\Delta_k = 1/k \sum_{i=1}^{k} x_i - \left(\prod_{i=1}^{k} x_i \right)^{1/k}.$$

If $n \geqq m$ prove that $n\Delta_n \geqq m\Delta_m$.

808. *Proposed by Anthony J. Strecok, Argonne National Laboratory.*

Evaluate the integral

$$I(x) = \int_0^{2\pi} e^{-x/\cos^2\theta}\, d\theta.$$

809. *Proposed by Furio Alberti, University of Illinois, Chicago Circle.*

It is shown in Muir, *A Treatise on the Theory of Determinants*, that the number of formal terms in the expansion of a zero axial determinant of order n is

$$T = n!\{1/2! - 1/3! + \cdots (-1)^n/n!\}, \qquad n = 2, 3, 4 \cdots$$

Show that $T = [n!/e + \{1 + (-1)^n\}/2]$ where the brackets denote the greatest integer function.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q523. Find the length of the longest sequence of nonzero digits in which an integral cube can terminate in the base 10.

[*Submitted by Miltiades S. Demos*]

Q524. If A and B are points on one branch of a hyperbola with foci P and Q' then P and Q are on one branch of a hyperbola with foci at A and B.

[*Submitted by William Wernick*]

Q525. Let f and g be differentiable functions on $[a, b]$ with $g'(x) \neq 0$ for all x in $[a, b]$. Prove that there exists c in $[a, b]$ such that

$$f'(c)/g'(c) = [f(c) - f(a)]/[g(b) - g(c)].$$

[*Submitted by Erwin Just*]

Q526. Find three three-digit prime numbers which together contain the nine positive digits and whose sum is a prime number also.

[*Submitted by Charles W. Trigg*]

Q527. Evaluate the determinant

$$D_n = |a_r - b_s|, \qquad r, s = 1, 2, \cdots, n.$$

[*Submitted by Murray S. Klamkin*]

(Answers on pages 239–240)

SOLUTIONS

Late Solutions

Robert G. Griswold, University of Hawaii, Hilo College: **770, 771**; *George A. Novacky, Jr., University of Pittsburgh:* **773**; *Charles W. Trigg, San Diego, California:* **773.**

Inverse Functions

775. [November, 1970] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Prove $\int_0^1 \sqrt[q]{1-x^p}\,dx = \int_0^1 \sqrt[p]{1-x^q}\,dx$, where $p, q > 0$.

I. *Solution by J. C. Binz, Bern, Switzerland.*

Let more generally f be a decreasing continuous function in $[a, b]$. Then the inverse function g exists in $[f(b), f(a)]$ and is also decreasing and continuous.

Compute

$$\int_{f(b)}^{f(a)} g(y)dy = \int_b^a g[f(t)]f'(t)dt = \int_b^a tf'(t)dt = af(a) - bf(b) + \int_a^b f(t)dt.$$

Hence, if additionally we have $f(a)=b$, $f(b)=a$, then $\int_a^b g(t)dt = \int_a^b f(t)dt.$

The functions $f: x \to \sqrt[p]{1-x^q}$ and $g: x \to \sqrt[q]{1-x^p}$ represent in $[0, 1]$ a special case of the preceding situation, which proves the proposition.

II. *Solution by Vàclav Konečný, Jarvis Christian College, Hawkins, Texas.*

$$\int_0^1 \sqrt[q]{1-x^p}\,dx = \frac{1}{p}\int_0^1 z^{-1+1/p}(1-z)^{1/q}dz$$
$$= \frac{1}{p}B(1/p, 1+1/q) = \frac{1}{pq}B(1/p, 1/q)$$

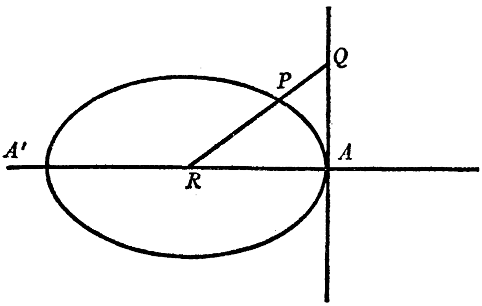
where p, $q>0$ to get the real value. We used the substitution $x^p=z$. B is the beta function and as $B(x, y)=B(y, x)$ the value of the integral is unchanged if we interchange p and q.

Also solved by Joseph Beer and Bernard August (jointly), Glassboro State College, New Jersey; Walter Blumberg, Flushing High School, New York; Dermott A. Breault, Cyber, Inc., Cambridge, Massachusetts; Robert X. Brennan, Dover, New Jersey; Robert J. Bridgman, Mansfield State College, Pennsylvania; David C. Brooks, Seattle Pacific College, Washington; G. R. Desai, St. Louis University; Robert Desko, Davenport, Iowa; Ellis Detwiler, Adams, New York; Santo M. Diano, Havertown, Pennsylvania; Fred Dodd, University of South Alabama; M. G. Greening, University of New South Wales, Australia; Robert G. Griswold, University of Hawaii, Hilo College; Philip Haverstick, Fort Belvoir, Virginia; Harry W. Hickey, Arlington, Virginia; John E. Homer, Lisle, Illinois; N. J. Kuenzi, Oshkosh, Wisconsin; David E. Mannes, SUNY, Oneonta, New York; Stephen B. Maurer, Phillips Exeter Academy; Edward Moylan, Ford Motor Company, Dearborn, Michigan; Albert J. Patsche, Rock Island Arsenal, Illinois; V. V. Ramana Rao, Andhra University, South India; B. E. Rhoades, Indiana University; Steve M. Rohde, General Motors Research Laboratories, Warren, Michigan; E. F. Schmeichel, College of Wooster, Ohio; Harry Siller, Hofstra University; A. Swyanavayanamuti, Andhra University, Waltair, South India; R. A. Struble, North Carolina State University; Philip Tracy, APO San Francisco; C. S. Venkataraman, Sree Kerala Varma College, Trichur, South India; John R. Ventura, Jr., Naval Underwater Systems Center, Newport, Rhode Island; R. L. Woodriff, Menlo College, Menlo Park, California; Thomas Wray, Department of Energy, Mines and Resources, Ottawa, Canada; Robert L. Young, Cape Cod Community College, Massachusetts; Paul Zwier, Calvin College, Michigan; and the proposer.

A Limiting Position

776. [November, 1970] *Proposed by Robert Siemann, University of Wisconsin, Waukesha.*

Given an ellipse as shown in the diagram. $\overleftrightarrow{AQ}$ is a tangent to it at one of its vertices A. Let P be a point on the ellipse such that $\overset{\frown}{AP}=\overleftrightarrow{AQ}$ corresponding to a point Q on the tangent AQ. Find the limiting position of R (the intersection of QP and axis AA') as P approaches A in the clockwise direction.



I. *Solution by James Russell, St. Procopius College, Illinois.*

Given:

$$\overset{\frown}{AP} = AQ = s.$$

$$A(a, 0) \qquad P(a\cos\theta, b\sin\theta) \qquad S(0, c)$$

$$Q(a, s) \qquad R(r, 0).$$

Equation of line PQ: $y = mx + c$

$$y = \left[\frac{s - b\sin\theta}{a - a\cos\theta}\right]x + c. \tag{1}$$

Since Q lies on this line we have

$$s = \left[\frac{s - b\sin\theta}{a - a\cos\theta}\right]a + c$$

or

$$c = \frac{b\sin\theta - s\cos\theta}{1 - \cos\theta}.$$

Hence (1) becomes

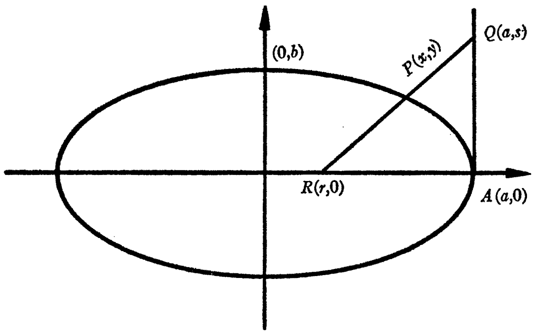
$$y = \left[\frac{s - b\sin\theta}{a(1 - \cos\theta)}\right]x + \frac{b\sin\theta - s\cos\theta}{1 - \cos\theta}. \tag{2}$$

To find R, consider that $(R, 0)$ is a point on this line.

$$R = \frac{a(s\cos\theta - b\sin\theta)}{s - b\sin\theta}. \tag{3}$$

To find $\lim_{\theta\to 0} R$, we must apply L'Hospital's rule three times to obtain

$$\lim_{\theta\to 0} R = \frac{a^2 - 3b^2}{a}. \tag{4}$$



II. *Solution by G. M. Rambousek, St. Procopius College, Illinois.*

Noting that R is on the line PQ, we have

$$\frac{y-0}{x-r} = \frac{s-y}{a-x}$$

or

$$r(x, y, s) = x + \frac{y(x-a)}{s-y}. \tag{1}$$

Now we want to obtain an expression for r in terms of a single variable and we choose ϕ, the eccentric angle of the ellipse, as the independent variable.

Since (x, y) is a point on the ellipse

$$x = a\cos\phi$$

$$y = b\sin\phi$$

$$\widehat{AP} = s = \int_0^\phi \sqrt{(dx)^2 + (dy)^2} = \int_0^\phi \sqrt{a^2\sin^2\phi + b^2\cos^2\phi}\; d\phi.$$

Hence,

$$r(\phi) = a\cos\phi + \frac{ab\sin\phi(\cos\phi - 1)}{\int_0^\phi \sqrt{a^2\sin^2\phi + b^2\cos\phi}\; d\phi - b\sin\phi}. \tag{2}$$

To find $\lim_{\phi\to 0} r(\phi)$, we must apply L'Hospital's rule three times yielding the value

$$a - \frac{3b^2}{a}.$$

Hence, the limiting position of R as P approaches a in the clockwise direction is

$$\left(a - \frac{3b^2}{a}, 0\right).$$

Also solved by Phil Adams, St. Procopius College; Walter Blumberg, Flushing High School, New York; Derrill J. Bordelon, Naval Underwater Systems Center, Newport, Rhode Island; Santo M. Diano, Havertown, Pennsylvania; Jorge Dou, Barcelona, Spain; Donnelly J. Johnson, Wright-Patterson Air Force Base, Ohio; J. F. Leetch, Bowling Green State University, Ohio; Vàclav Konecnỳ, Jarvis Christian College, Hawkins, Texas; Benjamin L. Schwartz, McLean, Virginia; Greg Swift, Omaha, Nebraska; Phil Tracy, APO San Francisco; E. T. Wong, Oberlin College; Thomas Wray, Department of Energy Mines and Resources, Ottawa, Canada; and R. L. Young, Cape Cod Community College, Massachusetts. Four incorrect solutions were received.

Pentagonal Inequality

777. [November, 1970] *Proposed by Alexandru Lupas, Institutul de Calcul, Cluj, Rumania.*

Let $P_5(A_1, A_2 \cdots A_5)$ be a convex pentagon and let $Q_5(B_1, B_2 \cdots B_5)$ be a convex pentagon which is determined by the line segments A_iA_{i+2}, $i=1, 2, \cdots, 5$ (e.g., $B_1 = A_1A_3 \cap A_5A_2$ and $B_5 = A_4A_1 \cap A_5A_2$). Let Ω be a point which is not in the exterior of Q_5. We denote by r_i the distances of Ω from the sides of P_5, d_i are the distances of Ω from the sides of Q_5 and $R_i = \Omega A_i$, $i=1, 2, \cdots, 5$. Prove that

$$3\sum_{i=1}^{5} R_i > 2\sum_{i=1}^{5} r_i + 4\sum_{i=1}^{5} d_i.$$

Solution by Stephen B. Maurer, Phillips Exeter Academy.

Consider one of the 5 triangles $A_iA_{i+2}A_{i+3}$. The point Ω is on or interior to this triangle. Thus, by the Erdos inequality, twice the sum of the distances from Ω to the sides of the triangle is equal or less than the sum of the distances to the vertices. But these latter three distances are all R_i's, whereas one of the former is an r_i and the other two are d_i's. Summing the inequality over all five triangles, we get the desired result.

Also solved by M. G. Greening, University of New South Wales, Australia; Jim Tattersall, Attleboro, Massachusetts; Phil Tracy, APO San Francisco; George B. Viau, Providence College, Rhode Island; and the proposer.

The Order of a Group

778. [November, 1970] *Proposed by Erwin Just, Bronx Community College.*

Let G be a finite group with subsets $T_1, T_2, \cdots, T_k$ chosen so that $|T_i| = 2^i$.

Prove that the smallest order of G for which it is possible that

$$G \neq \prod_{i=1}^{k} T_i \quad \text{is } 3\cdot 2^{k-1}.$$

Solution by Kevin Brown, Bowling Green State University.

We note first that if $S_1, \cdots S_n$ are nonempty subsets of a group then $|S_1 \cdots S_n| \geqq \max(|S_1|, \cdots |S_n|)$. To see this, fix all factors in the product $s_1 \cdots s_n$ except one, say s_i, and let this factor vary over the corresponding set S_i. This yields $|S_i|$ distinct elements of G. Now if $g \notin T_1 \cdots T_k$ then gT_k^{-1} and $T_1 \cdots T_{k-1}$ are disjoint subsets of G. We have $|gT_k^{-1}| = |T_k| = 2^k$ and $|T_1 \cdots T_{k-1}| \geqq |T_{k-1}| = 2^{k-1}$. Therefore $|G| \geqq 2^k + 2^{k-1} = 3\cdot 2^{k-1}$. For an example, let H be any group of order 2^k and let G be the direct product of H with a cyclic group $\langle b \rangle$ of order 3. Let $T_1 \cdots T_{k-1}$ be subsets of H so that $T_{k-1} = H$ and let $T_k = H \cup bH$. Then $T_1 \cdots T_k = T_k \neq G$.

Also solved by Stephen B. Maurer, Phillips Exeter Academy; E. F. Schmeichel, College of Wooster, Ohio; and the proposer.

A Trigonometric Inequality

779. [November, 1970] *Proposed by Sidney H. L. Kung, Jacksonville University, Florida.*

Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be real numbers. Show that $\prod_{j=1}^{n} |\sin^j \alpha_j \cos \alpha_j| \leqq 1/(n+1)^{(n+1)/2}$, and if $0 \leqq \alpha_j \leqq \pi/2$ the equality sign holds if and only if

$$\alpha_j = \cos^{-1} 1/\sqrt{j-1}, \qquad j = 1, 2, \cdots, n.$$

Solution by B. E. Rhoades, Indiana University.

Let $f(\alpha_1, \alpha_2 \cdots \alpha_n) = \prod_{j=1}^{n} \sin^j \alpha_j \cos \alpha_j$. Then $f(\alpha_1, \alpha_2, \cdots \alpha_n) = \prod_{j=1}^{n} f_j(\alpha_j)$, where $f_j(\alpha_j) = \sin^j \alpha_j \cos \alpha_j$. The extreme values of each f_j occur at points, call them $\bar{\alpha}_j$, which satisfy the equations $j \cos^2 \bar{\alpha}_j = \sin^2 \bar{\alpha}_j$. Moreover, $f_j(\bar{\alpha}_j) = \sin^{j+1} \bar{\alpha}_j / \sqrt{j}$. Thus

$$|f| \leqq \prod_{j=1}^{n} |f_j(\bar{\alpha}_j)| = \prod_{j=1}^{n} \frac{1}{\sqrt{j}} \frac{(j)^{(j+1)/2}}{j+1} = P(n),$$

say. By induction, one can show that

$$P(n) = 1/(n+1)^{(n+1)/2}.$$

If $0 \leqq \alpha_j \leqq \pi/2$, then we have equality if and only if f assumes its maximum: i.e., $\alpha_j = \cos^{-1} 1/\sqrt{j+1}$.

Also solved by Walter Blumberg, Flushing High School, New York; Wray G. Brady, Slippery Rock State College, Pennsylvania; E. M. Clarke, Jr., Madison College, Virginia; Santo M. Diano, Havertown, Pennsylvania; Ragnar Dybvik, Tingvoll, Norway; M. G. Greening, University of New South Wales, Australia; P. L. Hon, Victoria Technical School, Hong Kong; N. J. Kuenzi, Oshkosh, *Wisconsin; Doug McCallum, Eugene, Oregon; Albert J. Patsche, Rock Island Arsenal, Illinois; Steve M. Rohde, General Motors Research Laboratories, Warren, Michigan; E. F. Schmeichel, College of Wooster, Ohio; Bruce W. Tonkin, Oakland University, Michigan; Phil Tracy, APO San Francisco; John R. Ventura, Jr., Naval Underwater Systems Center, Newport, Rhode Island; and the proposer.*

A Maximal Convex Body

780. [November, 1970] *Proposed by Simeon Reich, Israel Institute of Technology, Haifa, Israel.*

Let there be given a plane bounded closed convex set with interior points and with boundary of length p. If $p<\pi(2+\sqrt{3})/3$, then one can rotate and translate this set in the plane so that in one position at least it will contain no lattice points.

Solution by E. F. Schmeichel, College of Wooster, Ohio.

The bound can be improved. L. G. Schnirelman has proved that on any closed curve, four points can be chosen forming the vertices of a square (Uspehi Mat. Nauk, Vol. X, 1944, pp. 34–44). If $p<4$ (cf. $\pi(2+\sqrt{3})/3 \doteq 3.91$), the side length of this square must be less than 1. So if we translate and rotate the figure until the square is centered at (1/2, 1/2) with sides parallel to the coordinate axes, the figure will not contain a lattice point.

Also solved by the proposer.

Quasi-Fermat

781. [November, 1970] *Proposed by Claude Raifaizen, Bayside, New York.*

If a, b, c, and n are integers and furthermore if $a^n+b^n=c^n$ with either a or b prime or the power of a prime and n an even positive integer, then n does not exceed two.

Solution by Charles W. Trigg, San Diego, California.

Without loss of generality, it may be assumed that a, b, and c have no common factor.

Let $n=2k$ and $a=p^r$. Then

$$p^{2rk} = c^{2k} - b^{2k} = (c^k + b^k)(c^k - b^k). \tag{1}$$

Consequently $c^k+b^k=p^s$ and $c^k-b^k=p^t$, where $s+t=2rk$. It follows that $2c^k=p^s+p^t$ and $2b^k=p^s-p^t$. Hence, $p|c$ and $p|b$, which is contrary to the assumption. The only situation in which (1) holds is when $c^k-b^k=1$. That is, $k=1$, $n=2$, and $c=b+1$.

The latter condition is evident otherwise when the two-parameter form of the sides of a primitive Pythagorean triangle is considered. Thus

$$(m^2 - n^2)^2 = (m^2 + n^2)^2 - (2mn)^2.$$

If $m^2-n^2=(m+n)(m-n)=p^r$, then $m=n+1$, and

$$a = m + n = 2n + 1, \quad b = 2n^2 + 2n, \quad c = 2n^2 + 2n + 1.$$

Also solved by Merrill Barnebey, Wisconsin State University at LaCrosse; J. C. Binz, Bern, Switzerland; Walter Blumberg, Flushing High School, New York; M. G. Greening, University of New South Wales, Australia; Ned Harrell, Menlo-Atherton High School, California; Heiko Harborth, Braunschweig, Germany; Erwin Just, Bronx Community College, New York; Lew Kowarski, Morgan State College, Maryland; George A. Novacky, Jr., University of Pittsburgh; Bob Prielipp and N. J. Kuenzi (jointly), Wisconsin State University at Oshkosh; E. F. Schmeichel, College of Wooster, Ohio; Phil Tracy, APO San Francisco; Kenneth M. Wilke, Topeka, Kansas; and the proposer.

Comment on Problem 84

84. [January, 1951, September, 1965, and September, 1966] *Proposed by Dewey Duncan, Los Angeles, California.*

We define a heterosquare as a square array of the first n^2 positive integers, so arranged that no two rows, columns, and diagonals (broken as well as straight) have the same sum.

(a) Show that no heterosquare of order two exists.

(b) Find a heterosquare of order three.

Comment by Charles W. Trigg, San Diego, California.

The sums of the elements of the rows, columns, and diagonals (broken and unbroken) of a 3-by-3 array constitute twelve triad sums of the square array. These sums are invariant under row and column transpositions, diagonal reflections and the shearing transformation,

$$\begin{matrix} a & h & f \\ d & b & k \\ g & e & c \end{matrix} \leftarrow \begin{matrix} a & b & c \\ d & e & f \\ g & h & k \end{matrix} \rightarrow \begin{matrix} a & b & c \\ f & d & e \\ h & k & g \end{matrix}$$

Usually, squares derived from a square by rotation or reflection are considered equivalent and not distinct from the original square. Thus the distinct squares derivable from row or column transposition may be found by cyclic permutation of the columns followed by cyclic permutation of the rows of the original and two derived squares. These nine squares constitute a *permutation group*.

Three successive horizontal (or vertical) shearing transformations regenerate the original square. Distinct squares derivable by this process may be found by applying both horizontal and vertical shearing, then to each derived square the other type of shearing, and again reversing the type applied to the squares thus derived. After removal of equivalents, 24 distinct squares (including the original one) remain. These fall by fours on six permutation groups, to produce 54 distinct squares having the same twelve triad sums, as in the table.

If these sums are distinct, the squares are *heterosquares*. If only one duplicate sum appears, the squares are *almost heterosquares*. Pinzka [1] and Nagara [2] have shown that if the elements are the nine positive digits, no heterosquares exist. Nagara found two almost heterosquares, complementary in that their correspondingly placed elements sum to 10.

The one of these squares which is in the upper right hand corner of the table determines 108 almost heterosquares—the 54 in the table and their complements. Each row of the table is a permutation group. The squares in the first, second, fourth, and fifth columns are those produced by the shearing transformation.

127	271	712	485	854	548	693	936	369
485	854	548	693	936	369	127	271	712
693	936	369	127	271	712	485	854	548
127	271	712	548	485	854	936	369	693
548	485	854	936	369	693	127	271	712
936	369	693	127	271	712	548	485	854
127	271	712	854	548	485	369	693	936
854	548	485	369	693	936	127	271	712
369	693	936	127	271	712	854	548	485
195	951	519	423	234	342	687	876	768
423	234	342	687	876	768	195	951	519
687	876	768	195	951	519	423	234	342
138	381	813	526	265	652	947	479	794
526	265	652	947	479	794	138	381	813
947	479	794	138	381	813	526	265	652
164	641	416	829	298	982	357	573	735
829	298	982	357	573	735	164	641	416
357	573	735	164	641	416	829	298	982

The twelve triad sums of these squares are 9, 10, 11, 12, 13, 15, 15, 17, 18, 19, 20, and 21. The duplicate sum is formed from $1+5+9$ and $3+5+7$ which occur in lines making a 45° angle. Hence, if neither of these triads appear in an unbroken diagonal, the square is *antimagic* with eight distinct sums for the rows, columns, and straight diagonals. The 37 antimagic squares in the table (which include all members of the fourth row) are designated by vertical lines on the right of the squares. These and their complements comprise the 74 antimagic squares determined by Nagara's squares.

References

1. C. F. Pinzka, Heterosquares, this MAGAZINE, 38 (1965) 250–252.
2. Prasert Na Nagara, Comment on Problem 84, this MAGAZINE, 39 (1966) 255–256.

Comment on Q485

Q485. [September, 1970] If concentric squares with parallel sides have areas in the ratio 2:1, then the segments drawn through the vertices of the smaller square perpendicular to the diagonals form with the segments of the larger square a regular octagon.

[*Submitted by Torquist Memp*]

Comment by Paul B. Johnson, University of California at Los Angeles.

Perhaps an easier solution is the following. In Figure 1, extend the segments to form Figure 2. The large squares are congruent since $PQ=ST=AB$, using the given area relationship. Symmetry shows the eight midsegments of the sides form a regular octagon.

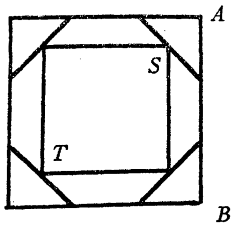


FIG. 1.

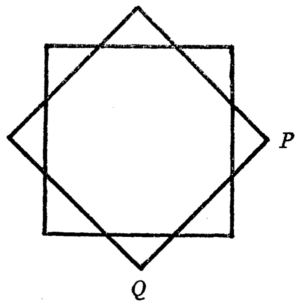


FIG. 2.

Comment on Q496

496. [January, 1971] Find two linearly independent functions whose Wronskians vanish identically.

[*Submitted by C. Stanley Ogilvy*]

Comment by Sid Spital, California State College at Hayward.

Another example is given by the independent functions x^2 and $x|x|$. Their Wronskian

$$W = \begin{vmatrix} x^2 & x|x| \\ 2x & 2|x| \end{vmatrix} = 0$$

for all x. In fact for any integer $n \geqq 1$, x^{n+1} and $x^n|x|$ would similarly work.

Comment on Q503

Q503. [January, 1971] A boy walks 4 mph, a girl walks 3 mph, and a dog walks 10 mph. They all start together at a certain place on a straight road, and the boy and girl walk steadily in the same direction. The dog walks back and forth between the two of them, going repeatedly from one to the other and back again. After one hour where is the dog and which direction is he facing?

[*Submitted by A. K. Austin, University of Sheffield*]

I. *Comment by M. S. Klamkin, Ford Motor Company.*

I disagree with the proposer's solution. While I agree that the motion is reversible from any initial starting position in which the three participants are not at the same location, it is not possible to start the motion when all three start from the same location. The dog would have a nervous breakdown attempting to carry out his program. If one is not convinced, let the initial starting distance between the boy and the girl be ϵ (arbitrarily small), then one can show that the number of times the dog reverses becomes arbitrarily large in a finite time.

An analogous situation occurs in the well known problem of the four bugs pursuing each other cyclically with the same constant speed and starting initially at the vertices of a square. At any point of their motion (except when together), the motion is reversible by reversing the velocities. However, when together, the directions of the velocities are indeterminate and thus they cannot reverse without further instructions.

II. *Comment by Leon Bankoff, Los Angeles, California.*

The published solution is based on the fallacious *a priori* assumption that the dog can be placed at a point between the positions of the boy and the girl. The conditions of the problem preclude the possibility of this occurrence. At any moment after the simultaneous start, the arrangement of the three will be girl, boy, dog—never girl, dog, boy.

III. *Comment by Charles W. Trigg, San Diego, California.*

At the end of one hour the dog is 10 miles from the starting point and facing away from it, since it never is between the boy and girl. That is, unless its position on the circle around the earth on which the straight road lies is considered. In that event it is between the boy and girl, leaving the boy behind and catching up with the girl. It will not need to change its direction for a long time, not until it has traversed the circumference and continuing on, passed the girl and caught up with the boy. They all should be well exhausted by that time.

IV. *Comment by Lyle E. Pursell, University of Missouri-Rolla.*

Quickie Q503 is self-contradictory. If the boy, girl and dog start from the same point at speeds of 4, 3 and 10, respectively, then at any *positive* time t, *no matter how small,* the dog will not be between the boy and the girl. Hence, the dog cannot run "back and forth between the two of them, going repeatedly from one to the other and back again" as the author prescribes.

The author's solution to the problem looks like a proposal to sum an infinite series by starting with the "last" term! Since, if later the three reverse their motion as the author suggests in his solution, then the dog must reverse his direction infinitely many times before the boy and the girl get back to the starting point.

ANSWERS

A523. The length is unbounded for $(10^k-1)^3=10^{3k}+3\cdot 10^{2k}+3\cdot 10^k-1$ has k terminal 9's where k is arbitrary. This is true for all odd powers.

A524. From the locus definition of a hyperbola, the hypothesis indicates that $AP-AQ=BP-BQ$, which implies $AP-BP=AQ-BQ$. That is $PA-PB=QA-QB$, which implies the conclusion.

A525. Define

$$h(x)=\frac{[f(x)-f(a)][g(x)-g(a)]}{g(b)-g(a)}-f(x)+f(a).$$

Since $g'(x)\neq 0$, $g(b)-g(a)\neq 0$. Now $f(x)$ is differentiable on $[a, b]$ and $h(a)=h(b)=0$. Therefore, from Rolle's theorem, it may be concluded that there exists a c in (a, b) such that $h'(c)=0$. The computation of $h'(c)$ and the hypothesis that $g'(x)\neq 0$ leads to the desired conclusion.

A526. If the three three-digit prime numbers together contain the nine positive digits, their sum $S\equiv 0$ so S can never be prime.

A527. Since D vanishes for $a_p=a_q$, $p\neq q$, and is linear in a_r, it must identically vanish for $n>2$. Also $D_1=a_1-b_1$ and $D_2=(a_1-a_2)(b_1-b_2)$.

(Quickies on page 229)

ANNOUNCEMENT OF LESTER R. FORD AWARDS

At its meeting on January 27, 1965, in Denver, Colorado, the Board of Governors authorized a number of awards, to be named after Lester R. Ford, Sr., to authors of expository articles published in the MONTHLY and the MATHEMATICS MAGAZINE. A maximum of six awards will be made annually; each award is in the amount of $100. The articles are to be selected by a subcommittee of the Committee on Publications appointed for this purpose.

The 1971 recipients of these Awards, selected by a committee consisting of Ivan Niven, Chairman, Marvin Marcus, and D. E. Richmond, were announced by President Klee at the business meeting of the Association on August 31, 1971, at Pennsylvania State University. The recipients of the Ford Awards for articles published in 1970 were the following:

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HENRY L. ALDER, *Secretary*

A524. From the locus definition of a hyperbola, the hypothesis indicates that $AP-AQ=BP-BQ$, which implies $AP-BP=AQ-BQ$. That is $PA-PB=QA-QB$, which implies the conclusion.

A525. Define

$$h(x)=\frac{[f(x)-f(a)][g(x)-g(a)]}{g(b)-g(a)}-f(x)+f(a).$$

Since $g'(x)\neq 0$, $g(b)-g(a)\neq 0$. Now $f(x)$ is differentiable on $[a, b]$ and $h(a)=h(b)=0$. Therefore, from Rolle's theorem, it may be concluded that there exists a c in (a, b) such that $h'(c)=0$. The computation of $h'(c)$ and the hypothesis that $g'(x)\neq 0$ leads to the desired conclusion.

A526. If the three three-digit prime numbers together contain the nine positive digits, their sum $S\equiv 0$ so S can never be prime.

A527. Since D vanishes for $a_p=a_q$, $p\neq q$, and is linear in a_r, it must identically vanish for $n>2$. Also $D_1=a_1-b_1$ and $D_2=(a_1-a_2)(b_1-b_2)$.

(Quickies on page 229)

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